CS265/CME309, Problem Set 5

SUNet ID(s):
Name(s):

By turning in this assignment, I agree by the Stanford honor code and declare that all of the writing is the work of my partner and I (discussion in larger groups is permissible).

Due by 11:59 PM on **Tuesday**, October 29th.

Theorem 1 *(Johnson and Lindenstrauss, 84') Given any $\epsilon \in (0,1)$, and a set $X \subset \mathcal{R}^k$ with $|X| = n$, there exists a randomized linear map $f : \mathcal{R}^k \to \mathcal{R}^d$ with $d = O(\frac{\log n}{\epsilon^2})$ that embeds (X, ℓ_2) into $(\mathcal{R}^d, \ell_2)$ so that with high probability*

$$(1-\epsilon)\|x-y\|_2^2 \leq \|f(x)-f(y)\|_2^2 \leq (1+\epsilon)\|x-y\|_2^2$$

for any $x, y \in X$.

1. (6 points) Prove that the logarithmic dependence on the number of points, n, in the above theorem cannot be improved. Specifically, define a set of n points $X \subset \mathcal{R}^k$ and some constant c such that it is impossible to embed them into $(\mathcal{R}^{c\log n}, \ell_2)$ without incurring distortion at least 2. Note that you are free to choose the dimension k to be anything you want! [Hint: Feel free to assume that a sphere in d dimensions of radius r has volume in the interval $[r^d/(d/2)^{d/2}, 2^{d/2}r^d/(d/2)^{d/2}]$. More specifically, come up with a set of pairwise distances that can be instantiated in $k = n$ dimensional space, but has the property that if an embedding into d dimensional space exists, then the image of all the points must lie within a ball of radius 1 in order to avoid having distortion more than 2. Now, leverage the fact that a sphere in d dimensions (for even d) of radius r has volume $\frac{\pi^{d/2}}{(d/2)!}r^d \in [r^d/(d/2)^{d/2}, 2^{d/2}r^d/(d/2)^{d/2}]$, to argue that there simply is not enough room in $d = 0.001\log n$ dimensions to accommodate the images of all the points without distorting some of their distances too much.]

   SOLUTION: [There are a few ways of doing this problem. Here is one..] Consider the n points in $\mathbf{R}^{n-1}$ dimension $n-1$ consisting of the $n-1$ unit-length basis vectors, $b_1, \ldots, b_{n-1}$ together with the all-zeros vector, b_0. The $n-1$ points have distance 1 from b_0, and distance $\sqrt{2}$ from each other. Consider some mapping $f : \mathbf{R}^{n-1} \to \mathbb{R}^{c\log n}$, for a constant c that we will specify later. Letting α denote $\max_{i\in\{1,\ldots,n-1\}} \|f(b_i) - f(b_0)\|_2$, it must hold that $f(b_0), \ldots, f(b_{n-1})$ all lie within the sphere of radius α centered at $f(b_0)$. We now argue that there is not enough volume in that sphere to accommodate all these points without two of them being too close together.

   If the map f has distortion < 2, then the distances between the images of $b_1, \ldots, b_{n-1}$ must be at least $\alpha\sqrt{2}/2$, which implies that, if one were to put spheres of radius $\alpha\sqrt{2}/4$ around each of these points, these spheres would be disjoint. Each of these spheres has volume at least $(\alpha\sqrt{2}/4)^{c\log n}/((c\log n)/2)^{(c\log n)/2}$, and hence the total volume of all these is at least

$$(n-1)(\alpha\sqrt{2}/4)^{c\log n}/((c\log n)/2)^{(c\log n)/2} = (n-1)\alpha^{c\log n}\left(\frac{1}{4c\log n}\right)^{(c\log n)/2}.$$

   All of these balls must also be contained within the ball of radius $\alpha + \alpha\sqrt{2}/4 < 2\alpha$ about $f(b_0)$. The radius of this ball, however, is *at most* $\alpha^{c\log n}16^{(c\log n)/2}$, using the formula from the hint. This is smaller than sum of the volumes of the small balls, if $n - 1 < (4\cdot 16)^{(c\log n)/2}$. If $c < \frac{1}{\log(4\cdot 16)}$, then the right side of this expression becomes $n^{1/2}$, which is less than $n-1$ for sufficiently large n (i.e. $n \geq 2$:)

2. In class we saw a very simple probabilistic argument that in any graph (V, E), the max-cut cuts at least $|E|/2$ edges. Here we consider two improvements/variants:

   (a) (4 points) Consider the greedy algorithm that, given an ordering $v_1, \ldots, v_n$ of the vertices, assigns v_1 to set A, then greedily partitions the other vertices (by sequentially assigning each unassigned vertex v to either A or B according to whether v has more neighbors already assigned to B or more neighbors already assigned to A.) Assume that ties are broken by assigning the point to set A. Prove that the cut found by this greedy algorithm cuts at least $\frac{|E|+|B|}{2}$ edges, where $|B|$ is the size of set B at the end of the algorithm. [Hint: think about what happens when a vertex actually gets put in set B.]

(b) (4 points) Prove that if the graph has an even number of nodes, n, then there exists a cut with at least $\frac{|E|}{2}\frac{n}{n-1}$ edges crossing the cut. [Hint: the fact that that n is even is just meant to suggest that it can be divided into two equal sets. . . .]

(c) **Double Bonus:** Can you turn the above probabilistic argument into an efficient deterministic algorithm? [This problem will not be graded...it is simply food for thought.]

3. (6 points) An *independent set* in a graph is a subset of vertices with the property that no pair of them are connected via an edge. Given a graph $G = (V, E)$ with $|V| = n$, prove that G has an independent set of size at least

$$\sum_{i=1}^{n} \frac{1}{d_i + 1},$$

where d_i is the degree of vertex v_i. [Hint: Given a permutation $\pi = v_{\pi(1)}, \ldots, v_{\pi(n)}$ of the vertices, define the corresponding (independent) set $S(\pi)$ by including vertex $v_{\pi(i)}$ if none of $v_{\pi(1)}, \ldots, v_{\pi(i-1)}$ is a neighbor of $v_{\pi(i)}$. Note that $S(\pi)$ is always an independent set. Now, define a randomized scheme for generating a permutation π, such that $\mathbf{E}[|S(\pi)|]$ is the claimed quantity above.]

4. Thursday night is halloween, and there is a student-faculty costume party mixer. Assume n people show up to the party: some of them are students, and the rest are faculty. Assume that each person has at least $1 + \log_2 n$ different costumes available to choose from (my set of costumes might or might not overlaps with your set). Note: its kindof funny if you show up with the same costume as your friends, but its horribly embarrassing if you have the same costume as one of the professors. Is it possible for all n people to coordinate their costumes in such a way that no student has the same costume as a professor? You will show, via the probabilistic method, that such a coordination is possible. hooray!

(a) (6 points) Recall that the probabilistic method is strictly BYOP (bring your own pumpkin, I mean probability distribution). Define a probabilistic scheme for choosing costumes, such that with positive probability, the chosen costumes satisfy the desired property (i.e. no student has the same costume as a professor). [Hint: try using a probability distribution over costumes that has some coordination between people. If each person selects one of their costumes uniformly at random from their set of $\geq 1 + \log_2 n$ costumes, then the resulting scheme does NOT seem to work.]

SOLUTION NOTE: In general, I try to avoid having questions on psets/exams that have a 1-line proof but take a long time to figure out. In this case, though, I quite like it, since it illustrates the magic of the probabilistic method: it might take a lot of effort to come up with the right distribution, but if you do come up with the right distribution, often the analysis ends up being super slick.