Part I: FOUNDATIONS

1. Propositional Logic(PL)

Propositional Logic(PL)

PL Syntax

Atom	truth symbols $\top$("true") and $\bot$("false")
	propositional variables $P, Q, R, P_1, Q_1, R_1, \cdots$
Literal	atom α or its negation $\neg\alpha$
Formula	literal or application of a
	logical connective to formulae F, F_1, F_2

$\neg F$	"not"	(negation)
$F_1 \wedge F_2$	"and"	(conjunction)
$F_1 \vee F_2$	"or"	(disjunction)
$F_1 \rightarrow F_2$	"implies"	(implication)
$F_1 \leftrightarrow F_2$	"if and only if"	(iff)

Example:

formula $F : (P \wedge Q) \rightarrow (\top \vee \neg Q)$
atoms: $P, Q, \top$
literal: $\neg Q$
subformulas: $P \wedge Q, \quad \top \vee \neg Q$
abbreviation
$F : P \wedge Q \rightarrow \top \vee \neg Q$

PL Semantics (meaning)

Sentence F + Interpretation I = Truth value (true, false)

Interpretation

$$I : \{P \mapsto \text{true}, Q \mapsto \text{false}, \cdots\}$$

Evaluation of F under I:

F	$\neg F$
0	1
1	0

where 0 corresponds to value false
1 true

F_1	F_2	$F_1 \wedge F_2$	$F_1 \vee F_2$	$F_1 \rightarrow F_2$	$F_1 \leftrightarrow F_2$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Example:

$F : P \land Q \rightarrow P \lor \neg Q$
$I : \{P \mapsto \text{true}, Q \mapsto \text{false}\}$

P	Q	$\neg Q$	$P \land Q$	$P \lor \neg Q$	F
1	0	1	0	1	1

$1 = \text{true}$ $\quad$ $0 = \text{false}$

F evaluates to true under I

Inductive Definition of PL's Semantics

$I \models F$ if F evaluates to true under I
$I \not\models F$ false

Base Case:

$I \models \top$
$I \not\models \bot$

$I \models P$ iff $I[P] = \text{true}$
$I \not\models P$ iff $I[P] = \text{false}$

Inductive Case:

$I \models \neg F$ iff $I \not\models F$
$I \models F_1 \land F_2$ iff $I \models F_1$ and $I \models F_2$
$I \models F_1 \lor F_2$ iff $I \models F_1$ or $I \models F_2$
$I \models F_1 \rightarrow F_2$ iff, if $I \models F_1$ then $I \models F_2$
$I \models F_1 \leftrightarrow F_2$ iff, $I \models F_1$ and $I \models F_2$,
or $I \not\models F_1$ and $I \not\models F_2$

Note:

$I \not\models F_1 \rightarrow F_2$ iff $I \models F_1$ and $I \not\models F_2$

Example:

$$F : P \land Q \rightarrow P \lor \neg Q$$

$$I : \{P \mapsto \text{true}, Q \mapsto \text{false}\}$$

1. $I \models P$ since $I[P] = \text{true}$
2. $I \not\models Q$ since $I[Q] = \text{false}$
3. $I \models \neg Q$ by 2 and $\neg$
4. $I \not\models P \land Q$ by 2 and $\land$
5. $I \models P \lor \neg Q$ by 1 and $\lor$
6. $I \models F$ by 4 and $\rightarrow$ Why?

Thus, F is true under I.

Satisfiability and Validity

F satisfiable iff there exists an interpretation I such that $I \models F$.
F valid iff for all interpretations I, $I \models F$.

F is valid iff $\neg F$ is unsatisfiable

Method 1: Truth Tables

Example $F : P \land Q \rightarrow P \lor \neg Q$

P	Q	$P \land Q$	$\neg Q$	$P \lor \neg Q$	F
0	0	0	1	1	1
0	1	0	0	0	1
1	0	0	1	1	1
1	1	1	0	1	1

Thus F is valid.

Example $F : P \lor Q \rightarrow P \land Q$

P	Q	$P \lor Q$	$P \land Q$	F	
0	0	0	0	1	← satisfying I
0	1	1	0	0	← falsifying I
1	0	1	0	0	
1	1	1	1	1	

Thus F is satisfiable, but invalid.

Method 2: Semantic Argument

Proof rules

$$\frac{I \models \neg F}{I \not\models F} \qquad \frac{I \not\models \neg F}{I \models F}$$

$$\frac{I \models F \land G}{\begin{array}{l} I \models F \\ I \models G \end{array}} \leftarrow \text{and} \qquad \frac{I \not\models F \land G}{I \not\models F \mid I \not\models G} \nwarrow \text{or}$$

$$\frac{I \models F \lor G}{I \models F \mid I \models G} \qquad \frac{I \not\models F \lor G}{\begin{array}{l} I \not\models F \\ I \not\models G \end{array}}$$

$$\frac{I \models F \rightarrow G}{I \not\models F \mid I \models G} \qquad \frac{I \not\models F \rightarrow G}{\begin{array}{l} I \models F \\ I \not\models G \end{array}}$$

$$\frac{I \models F \leftrightarrow G}{I \models F \land G \mid I \not\models F \lor G} \qquad \frac{I \not\models F \leftrightarrow G}{I \models F \land \neg G \mid I \models \neg F \land G}$$

$$\frac{\begin{array}{l} I \models F \\ I \not\models F \end{array}}{I \models \bot}$$

Example 1: Prove

$$F : P \land Q \rightarrow P \lor \neg Q \quad \text{is valid.}$$

Let's assume that F is not valid and that I is a falsifying interpretation.

1.	I	$\not\models$	$P \land Q \rightarrow P \lor \neg Q$	assumption
2.	I	$\models$	$P \land Q$	1 and $\rightarrow$
3.	I	$\not\models$	$P \lor \neg Q$	1 and $\rightarrow$
4.	I	$\models$	P	2 and $\land$
5.	I	$\not\models$	P	3 and $\lor$
6.	I	$\models$	$\bot$	4 and 5 are contradictory

Thus F is valid.

Example 2: Prove

$$F : (P \rightarrow Q) \land (Q \rightarrow R) \rightarrow (P \rightarrow R) \quad \text{is valid.}$$

Let's assume that F is not valid.

1.	I	$\not\models$	F	assumption
2.	I	$\models$	$(P \rightarrow Q) \land (Q \rightarrow R)$	1 and $\rightarrow$
3.	I	$\not\models$	$P \rightarrow R$	1 and $\rightarrow$
4.	I	$\models$	P	3 and $\rightarrow$
5.	I	$\not\models$	R	3 and $\rightarrow$
6.	I	$\models$	$P \rightarrow Q$	2 and of $\land$
7.	I	$\models$	$Q \rightarrow R$	2 and of $\land$

Two cases from 6

$8a.$	I	$\not\models$	P	6 and $\rightarrow$
$9a.$	I	$\models$	$\bot$	4 and 8a are contradictory

and

$8b.$	I	$\models$	Q	6 and $\rightarrow$

Two cases from 7

$9ba.$	I	$\not\models$	Q	7 and $\rightarrow$
$10ba.$	I	$\models$	$\bot$	8b and 9ba are contradictory

and

$9bb.$	I	$\models$	R	7 and $\rightarrow$
$10bb.$	I	$\models$	$\bot$	5 and 9bb are contradictory

Our assumption is incorrect in all cases — F is valid.

Example 3: Is

$$F:\ P \vee Q \rightarrow P \wedge Q \quad \text{valid?}$$

Let's assume that F is not valid.

1.	I	$\not\models$	$P \vee Q \rightarrow P \wedge Q$	assumption
2.	I	$\models$	$P \vee Q$	1 and $\rightarrow$
3.	I	$\not\models$	$P \wedge Q$	1 and $\rightarrow$

Two options

$4a.$	I	$\models$	P	2 and $\vee$	$4b.$	I	$\models$	Q	2 and $\vee$
$5a.$	I	$\not\models$	Q	3 and $\wedge$	$5b.$	I	$\not\models$	P	3 and $\wedge$

We cannot derive a contradiction. F is not valid.

Falsifying interpretation:
$I_1 :\ \{P \mapsto \text{true},\ Q \mapsto \text{false}\}$ $\quad I_2 :\ \{Q \mapsto \text{true},\ P \mapsto \text{false}\}$
We have to derive a contradiction in both cases for F to be valid.

Equivalence

F_1 and F_2 are equivalent ($F_1 \Leftrightarrow F_2$)
iff for all interpretations I, $I \models F_1 \leftrightarrow F_2$

To prove $F_1 \Leftrightarrow F_2$ show $F_1 \leftrightarrow F_2$ is valid.

F_1 implies F_2 ($F_1 \Rightarrow F_2$)
iff for all interpretations I, $I \models F_1 \rightarrow F_2$

$F_1 \Leftrightarrow F_2$ and $F_1 \Rightarrow F_2$ are not formulae!

Normal Forms

1. Negation Normal Form (NNF)
   Negations appear only in literals. (only $\neg$, $\wedge$, $\vee$)
   To transform F to equivalent F' in NNF use recursively the following template equivalences (left-to-right):

$$\neg\neg F_1 \Leftrightarrow F_1 \qquad \neg\top \Leftrightarrow \bot \qquad \neg\bot \Leftrightarrow \top$$

$$\left.\begin{array}{l} \neg(F_1 \wedge F_2) \Leftrightarrow \neg F_1 \vee \neg F_2 \\ \neg(F_1 \vee F_2) \Leftrightarrow \neg F_1 \wedge \neg F_2 \end{array}\right\} \text{De Morgan's Law}$$

$$F_1 \rightarrow F_2 \Leftrightarrow \neg F_1 \vee F_2$$

$$F_1 \leftrightarrow F_2 \Leftrightarrow (F_1 \rightarrow F_2) \wedge (F_2 \rightarrow F_1)$$

Example: Convert $F:\ \neg(P \rightarrow \neg(P \wedge Q))$ to NNF

$F' : \neg(\neg P \vee \neg(P \wedge Q))$	$\rightarrow$ to $\vee$
$F'' : \neg\neg P \wedge \neg\neg(P \wedge Q)$	De Morgan's Law
$F''' : P \wedge P \wedge Q$	$\neg\neg$

F''' is equivalent to F ($F''' \Leftrightarrow F$) and is in NNF

2. Disjunctive Normal Form (DNF)

Disjunction of conjunctions of literals

$$\bigvee_i \bigwedge_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

To convert F into equivalent F' in DNF,
transform F into NNF and then
use the following template equivalences (left-to-right):

$$\left.\begin{array}{lcl} (F_1 \lor F_2) \land F_3 & \Leftrightarrow & (F_1 \land F_3) \lor (F_2 \land F_3) \\ F_1 \land (F_2 \lor F_3) & \Leftrightarrow & (F_1 \land F_2) \lor (F_1 \land F_3) \end{array}\right\} \textit{dist}$$

Example: Convert

$$F : (Q_1 \lor \neg\neg Q_2) \land (\neg R_1 \to R_2) \text{ into DNF}$$

F'	$: (Q_1 \lor Q_2) \land (R_1 \lor R_2)$	in NNF
F''	$: (Q_1 \land (R_1 \lor R_2)) \lor (Q_2 \land (R_1 \lor R_2))$	dist
F'''	$: (Q_1 \land R_1) \lor (Q_1 \land R_2) \lor (Q_2 \land R_1) \lor (Q_2 \land R_2)$	dist

F''' is equivalent to F ($F''' \Leftrightarrow F$) and is in DNF

3. Conjunctive Normal Form (CNF)

Conjunction of disjunctions of literals

$$\bigwedge_i \bigvee_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

To convert F into equivalent F' in CNF,
transform F into NNF and then
use the following template equivalences (left-to-right):

$$\begin{array}{lcl} (F_1 \land F_2) \lor F_3 & \Leftrightarrow & (F_1 \lor F_3) \land (F_2 \lor F_3) \\ F_1 \lor (F_2 \land F_3) & \Leftrightarrow & (F_1 \lor F_2) \land (F_1 \lor F_3) \end{array}$$

Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Decides the satisfiability of PL formulae in CNF

In book, efficient conversion of F to F' where

F' is in CNF and
F' and F are equisatisfiable (F is satisfiable iff F' is satisfiable)

Decision Procedure DPLL: Given F in CNF

DPLL(F) =
  if $F = \top$ then true ··· F is satisfiable
  else if $F = \bot$ then false ··· F is unsatisfiable
  else if some P appears only positively then DPLL($F\{P \mapsto \top\}$)
  else if some P appears only negatively then DPLL($F\{P \mapsto \bot\}$)
  else
    let $F' = \text{BCP}(F)$ in
    if $F' \neq F$ then DPLL(F')
    else
      let $P = \text{choose}(\text{vars}(F))$ in
      DPLL($F\{P \mapsto \top\}$) $\lor$ DPLL($F\{P \mapsto \bot\}$)

Boolean Constraint Propagation (BCP)

Based on unit resolution

$$\frac{\ell \qquad C[\neg\ell]}{C[\bot]} \leftarrow \text{clause} \qquad \textit{where } \ell = P \textit{ or } \ell = \neg P$$

throughout

Example:

$$F : (\neg P \lor Q \lor R) \land (\neg Q \lor R) \land (\neg Q \lor \neg R) \land (P \lor \neg Q \lor \neg R)$$

Branching on Q

$$F\{Q \mapsto \top\} : (R) \land (\neg R) \land (P \lor \neg R)$$

By unit resolution

$$\frac{R \qquad (\neg R)}{\bot}$$

$F\{Q \mapsto \top\} = \bot \Rightarrow$ false

On the other branch

$F\{Q \mapsto \bot\} : (\neg P \lor R)$

$F\{Q \mapsto \bot,\ R \mapsto \top,\ P \mapsto \bot\} = \top \Rightarrow \text{true}$

F is satisfiable with satisfying interpretation

$$I : \{P \mapsto \text{false},\ Q \mapsto \text{false},\ R \mapsto \text{true}\}$$