

Part I: FOUNDATIONS

1. Propositional Logic(PL)

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PL Syntax

<u>Atom</u>	<u>truth symbols</u> \top (“true”) and \perp (“false”) <u>propositional variables</u> $P, Q, R, P_1, Q_1, R_1, \dots$
<u>Literal</u>	atom α or its negation $\neg\alpha$
<u>Formula</u>	literal or application of a <u>logical connective</u> to formulae F, F_1, F_2
	$\neg F$ “not” (negation)
	$F_1 \wedge F_2$ “and” (conjunction)
	$F_1 \vee F_2$ “or” (disjunction)
	$F_1 \rightarrow F_2$ “implies” (implication)
	$F_1 \leftrightarrow F_2$ “if and only if” (iff)

Example:

formula $F : (P \wedge Q) \rightarrow (T \vee \neg Q)$

atoms: P, Q, T

literal: $\neg Q$

subformulas: $P \wedge Q, T \vee \neg Q$

abbreviation

$$F : P \wedge Q \rightarrow T \vee \neg Q$$

PL Semantics (meaning)

Sentence F + Interpretation I = Truth value
(true, false)

Interpretation

$$I : \{P \mapsto \text{true}, Q \mapsto \text{false}, \dots\}$$

Evaluation of F under I :

F	$\neg F$	
0	1	where 0 corresponds to value false 1 true
1	0	

F_1	F_2	$F_1 \wedge F_2$	$F_1 \vee F_2$	$F_1 \rightarrow F_2$	$F_1 \leftrightarrow F_2$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Example:

$$F : P \wedge Q \rightarrow P \vee \neg Q$$

$$I : \{P \mapsto \text{true}, Q \mapsto \text{false}\}$$

P	Q	$\neg Q$	$P \wedge Q$	$P \vee \neg Q$	F
1	0	1	0	1	1

1 = true

0 = false

F evaluates to true under I

Inductive Definition of PL's Semantics

$I \models F$ if F evaluates to true under I
 $I \not\models F$ false

Base Case:

$I \models \top$

$I \not\models \perp$

$I \models P$ iff $I[P] = \text{true}$

$I \not\models P$ iff $I[P] = \text{false}$

Inductive Case:

$I \models \neg F$ iff $I \not\models F$

$I \models F_1 \wedge F_2$ iff $I \models F_1$ and $I \models F_2$

$I \models F_1 \vee F_2$ iff $I \models F_1$ or $I \models F_2$

$I \models F_1 \rightarrow F_2$ iff, if $I \models F_1$ then $I \models F_2$

$I \models F_1 \leftrightarrow F_2$ iff, $I \models F_1$ and $I \models F_2$,
or $I \not\models F_1$ and $I \not\models F_2$

Note:

$I \not\models F_1 \rightarrow F_2$ iff $I \models F_1$ and $I \not\models F_2$

Example:

$$F : P \wedge Q \rightarrow P \vee \neg Q$$

$$I : \{P \mapsto \text{true}, Q \mapsto \text{false}\}$$

1. $I \models P$ since $I[P] = \text{true}$
2. $I \not\models Q$ since $I[Q] = \text{false}$
3. $I \models \neg Q$ by 2 and \neg
4. $I \not\models P \wedge Q$ by 2 and \wedge
5. $I \models P \vee \neg Q$ by 1 and \vee
6. $I \models F$ by 4 and \rightarrow Why?

Thus, F is true under I .

Satisfiability and Validity

F satisfiable iff there exists an interpretation I such that $I \models F$.

F valid iff for all interpretations I , $I \models F$.

F is valid iff $\neg F$ is unsatisfiable

Method 1: Truth Tables

Example $F : P \wedge Q \rightarrow P \vee \neg Q$

P	Q	$P \wedge Q$	$\neg Q$	$P \vee \neg Q$	F
0	0	0	1	1	1
0	1	0	0	0	1
1	0	0	1	1	1
1	1	1	0	1	1

Thus F is valid.

Example $F : P \vee Q \rightarrow P \wedge Q$

P	Q	$P \vee Q$	$P \wedge Q$	F
0	0	0	0	1
0	1	1	0	0
1	0	1	0	0
1	1	1	1	1

← satisfying /

← falsifying /

Thus F is satisfiable, but invalid.

Method 2: Semantic Argument

Proof rules

$$\frac{I \models \neg F}{I \not\models F}$$

$$\frac{I \not\models \neg F}{I \models F}$$

$$\frac{I \models F \wedge G}{\begin{array}{l} I \models F \\ I \models G \end{array}}{\leftarrow \text{and}}$$

$$\frac{I \not\models F \wedge G}{\begin{array}{l} I \not\models F \\ I \not\models G \end{array}}{\leftarrow \text{or}}$$

$$\frac{I \models F \vee G}{I \models F \mid I \models G}$$

$$\frac{I \not\models F \vee G}{\begin{array}{l} I \not\models F \\ I \not\models G \end{array}}$$

$$\frac{I \models F \rightarrow G}{I \not\models F \mid I \models G}$$

$$\frac{I \not\models F \rightarrow G}{\begin{array}{l} I \models F \\ I \not\models G \end{array}}$$

$$\frac{I \models F \leftrightarrow G}{I \models F \wedge G \mid I \not\models F \vee G}$$

$$\frac{I \not\models F \leftrightarrow G}{I \models F \wedge \neg G \mid I \models \neg F \wedge G}$$

$$\frac{\begin{array}{l} I \models F \\ I \not\models F \end{array}}{I \models \perp}$$

Example 1: Prove

$F : P \wedge Q \rightarrow P \vee \neg Q$ is valid.

Let's assume that F is not valid and that I is a falsifying interpretation.

1. $I \not\models P \wedge Q \rightarrow P \vee \neg Q$ assumption
2. $I \models P \wedge Q$ 1 and \rightarrow
3. $I \not\models P \vee \neg Q$ 1 and \rightarrow
4. $I \models P$ 2 and \wedge
5. $I \not\models P$ 3 and \vee
6. $I \models \perp$ 4 and 5 are contradictory

Thus F is valid.

Example 2: Prove

$F : (P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$ is valid.

Let's assume that F is not valid.

- | | | | | |
|----|-----|---------------|--|---------------------|
| 1. | I | $\not\models$ | F | assumption |
| 2. | I | \models | $(P \rightarrow Q) \wedge (Q \rightarrow R)$ | 1 and \rightarrow |
| 3. | I | $\not\models$ | $P \rightarrow R$ | 1 and \rightarrow |
| 4. | I | \models | P | 3 and \rightarrow |
| 5. | I | $\not\models$ | R | 3 and \rightarrow |
| 6. | I | \models | $P \rightarrow Q$ | 2 and of \wedge |
| 7. | I | \models | $Q \rightarrow R$ | 2 and of \wedge |

Two cases from 6

8a. $I \not\models P$ 6 and \rightarrow

9a. $I \models \perp$ 4 and 8a are contradictory

and

8b. $I \models Q$ 6 and \rightarrow

Two cases from 7

9ba. $I \not\models Q$ 7 and \rightarrow

10ba. $I \models \perp$ 8b and 9ba are contradictory

and

9bb. $I \models R$ 7 and \rightarrow

10bb. $I \models \perp$ 5 and 9bb are contradictory

Our assumption is incorrect in all cases — F is valid.

Example 3: Is

$$F : P \vee Q \rightarrow P \wedge Q \quad \text{valid?}$$

Let's assume that F is not valid.

1. $I \not\models P \vee Q \rightarrow P \wedge Q$ assumption
2. $I \models P \vee Q$ 1 and \rightarrow
3. $I \not\models P \wedge Q$ 1 and \rightarrow

Two options

- | | | | |
|-----------------------|----------------|-----------------------|----------------|
| 4a. $I \models P$ | 2 and \vee | 4b. $I \models Q$ | 2 and \vee |
| 5a. $I \not\models Q$ | 3 and \wedge | 5b. $I \not\models P$ | 3 and \wedge |

We cannot derive a contradiction. F is not valid.

Falsifying interpretation:

$$I_1 : \{P \mapsto \text{true}, Q \mapsto \text{false}\} \quad I_2 : \{Q \mapsto \text{true}, P \mapsto \text{false}\}$$

We have to derive a contradiction in both cases for F to be valid.

Equivalence

F_1 and F_2 are equivalent ($F_1 \Leftrightarrow F_2$)

iff for all interpretations I , $I \models F_1 \leftrightarrow F_2$

To prove $F_1 \Leftrightarrow F_2$ show $F_1 \leftrightarrow F_2$ is valid.

F_1 implies F_2 ($F_1 \Rightarrow F_2$)

iff for all interpretations I , $I \models F_1 \rightarrow F_2$

$F_1 \Leftrightarrow F_2$ and $F_1 \Rightarrow F_2$ are not formulae!

Normal Forms

1. Negation Normal Form (NNF)

Negations appear only in literals. (only \neg , \wedge , \vee)

To transform F to equivalent F' in NNF use recursively the following template equivalences (left-to-right):

$$\begin{aligned} \neg\neg F_1 &\Leftrightarrow F_1 & \neg\top &\Leftrightarrow \perp & \neg\perp &\Leftrightarrow \top \\ \neg(F_1 \wedge F_2) &\Leftrightarrow \neg F_1 \vee \neg F_2 & & & & \\ \neg(F_1 \vee F_2) &\Leftrightarrow \neg F_1 \wedge \neg F_2 & & & & \\ & & & & & \left. \vphantom{\begin{aligned} \neg(F_1 \wedge F_2) \\ \neg(F_1 \vee F_2) \end{aligned}} \right\} \text{De Morgan's Law} \\ F_1 \rightarrow F_2 &\Leftrightarrow \neg F_1 \vee F_2 \\ F_1 \leftrightarrow F_2 &\Leftrightarrow (F_1 \rightarrow F_2) \wedge (F_2 \rightarrow F_1) \end{aligned}$$

Example: Convert $F : \neg(P \rightarrow \neg(P \wedge Q))$ to NNF

$$\begin{aligned} F' &: \neg(\neg P \vee \neg(P \wedge Q)) && \rightarrow \text{to } \vee \\ F'' &: \neg\neg P \wedge \neg\neg(P \wedge Q) && \text{De Morgan's Law} \\ F''' &: P \wedge P \wedge Q && \neg\neg \end{aligned}$$

F''' is equivalent to F ($F''' \Leftrightarrow F$) and is in NNF

2. Disjunctive Normal Form (DNF)

Disjunction of conjunctions of literals

$$\bigvee_i \bigwedge_j l_{i,j} \quad \text{for literals } l_{i,j}$$

To convert F into equivalent F' in DNF,
transform F into NNF and then

use the following template equivalences (left-to-right):

$$\left. \begin{aligned} (F_1 \vee F_2) \wedge F_3 &\Leftrightarrow (F_1 \wedge F_3) \vee (F_2 \wedge F_3) \\ F_1 \wedge (F_2 \vee F_3) &\Leftrightarrow (F_1 \wedge F_2) \vee (F_1 \wedge F_3) \end{aligned} \right\} \text{dist}$$

Example: Convert

$F : (Q_1 \vee \neg\neg Q_2) \wedge (\neg R_1 \rightarrow R_2)$ into DNF

$F' : (Q_1 \vee Q_2) \wedge (R_1 \vee R_2)$ in NNF

$F'' : (Q_1 \wedge (R_1 \vee R_2)) \vee (Q_2 \wedge (R_1 \vee R_2))$ dist

$F''' : (Q_1 \wedge R_1) \vee (Q_1 \wedge R_2) \vee (Q_2 \wedge R_1) \vee (Q_2 \wedge R_2)$ dist

F''' is equivalent to F ($F''' \Leftrightarrow F$) and is in DNF

3. Conjunctive Normal Form (CNF)

Conjunction of disjunctions of literals

$$\bigwedge_i \bigvee_j l_{i,j} \quad \text{for literals } l_{i,j}$$

To convert F into equivalent F' in CNF,
transform F into NNF and then
use the following template equivalences (left-to-right):

$$\begin{aligned}(F_1 \wedge F_2) \vee F_3 &\Leftrightarrow (F_1 \vee F_3) \wedge (F_2 \vee F_3) \\ F_1 \vee (F_2 \wedge F_3) &\Leftrightarrow (F_1 \vee F_2) \wedge (F_1 \vee F_3)\end{aligned}$$

Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Decides the satisfiability of PL formulae in CNF

In book, efficient conversion of F to F' where

F' is in CNF and

F' and F are equisatisfiable (F is satisfiable iff F' is satisfiable)

Decision Procedure DPLL: Given F in CNF

DPLL(F) =

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if  $F = \top$  then true           ...  $F$  is satisfiable
else if  $F = \perp$  then false     ...  $F$  is unsatisfiable
else if some  $P$  appears only positively then DPLL( $F\{P \mapsto \top\}$ )
else if some  $P$  appears only negatively then DPLL( $F\{P \mapsto \perp\}$ )
else
  let  $F' = \text{BCP}(F)$  in
  if  $F' \neq F$  then DPLL( $F'$ )
  else
    let  $P = \text{choose}(\text{vars}(F))$  in
    DPLL( $F\{P \mapsto \top\}$ )  $\vee$  DPLL( $F\{P \mapsto \perp\}$ )
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Boolean Constraint Propagation (BCP)

Based on unit resolution

$$\frac{\ell \quad C[\neg\ell]}{C[\perp]} \leftarrow \text{clause} \quad \text{where } \ell = P \text{ or } \ell = \neg P$$

throughout

Example:

$$F : (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R)$$

Branching on Q

$$F\{Q \mapsto \top\} : (R) \wedge (\neg R) \wedge (P \vee \neg R)$$

By unit resolution

$$\frac{R \quad (\neg R)}{\perp}$$

$$F\{Q \mapsto \top\} = \perp \Rightarrow \text{false}$$

On the other branch

$$F\{Q \mapsto \perp\} : (\neg P \vee R)$$

$$F\{Q \mapsto \perp, R \mapsto \top, P \mapsto \perp\} = \top \Rightarrow \text{true}$$

F is satisfiable with satisfying interpretation

$$I : \{P \mapsto \text{false}, Q \mapsto \text{false}, R \mapsto \text{true}\}$$

