# 10. Combining Decision Procedures

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## Combining Decision Procedures: Nelson-Oppen Method

**Given**

Theories $T_i$ over *signatures* $\Sigma_i$
(constants, functions, predicates)
with corresponding decision procedures $P_i$ for $T_i$-satisfiability.

**Goal**

Decide satisfiability of a sentence in theory $\cup_i T_i$.

**Example**: How do we show that

$$F: \; 1 \leq x \;\wedge\; x \leq 2 \;\wedge\; f(x) \neq f(1) \;\wedge\; f(x) \neq f(2)$$

is $(T_E \cup T_{\mathbb{Z}})$-unsatisfiable?

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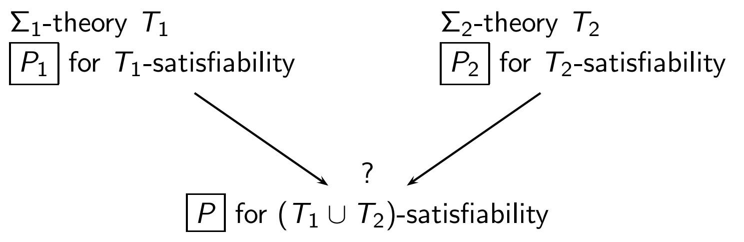
## Combining Decision Procedures



$\Sigma_1$-theory $T_1$
$\boxed{P_1}$ for $T_1$-satisfiability

$\Sigma_2$-theory $T_2$
$\boxed{P_2}$ for $T_2$-satisfiability

?

$\boxed{P}$ for $(T_1 \cup T_2)$-satisfiability

**Problem**:
Decision procedures are domain specific.
How do we combine them?

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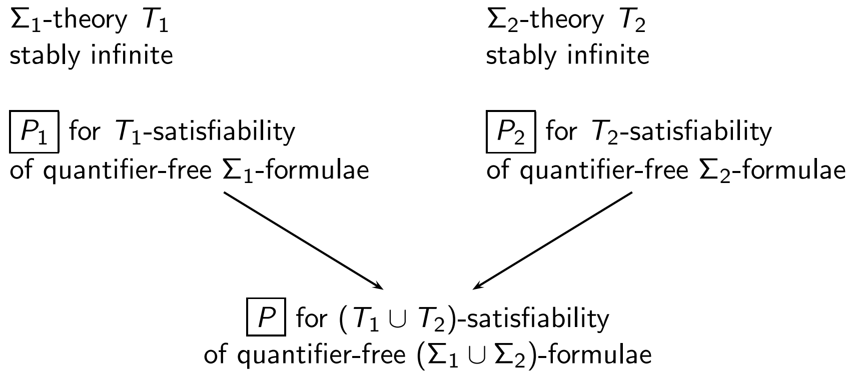
## Nelson-Oppen Combination Method (N-O Method)

$$\Sigma_1 \cap \Sigma_2 = \emptyset$$



$\Sigma_1$-theory $T_1$
stably infinite

$\Sigma_2$-theory $T_2$
stably infinite

$\boxed{P_1}$ for $T_1$-satisfiability
of quantifier-free $\Sigma_1$-formulae

$\boxed{P_2}$ for $T_2$-satisfiability
of quantifier-free $\Sigma_2$-formulae

$\boxed{P}$ for $(T_1 \cup T_2)$-satisfiability
of quantifier-free $(\Sigma_1 \cup \Sigma_2)$-formulae

## Nelson-Oppen: Limitations

Given formula $F$ in theory $T_1 \cup T_2$.

1. $F$ must be quantifier-free.
2. Signatures $\Sigma_i$ of the combined theory only share $=$, i.e.,
$$\Sigma_1 \cap \Sigma_2 = \{=\}$$
3. Theories must be stably infinite.

Note:

- Algorithm can be extended to combine arbitrary number of theories $T_i$ — combine two, then combine with another, and so on.
- We restrict $F$ to be conjunctive formula — otherwise convert to DNF and check each disjunct.

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## Stably Infinite Theories

A $\Sigma$-theory $T$ is *stably infinite* iff
for every quantifier-free $\Sigma$-formula $F$:
if $F$ is $T$-satisfiable
then there exists some $T$-interpretation that satisfies $F$.

**Example:** $\Sigma$-theory $T$
$$\Sigma : \{a, b, =\}$$

Axiom

$$\forall x.\ x = a \ \vee \ x = b$$

For every $T$-interpretation $I$, $|D_I| \leq 2$ (at most two elements).
Hence, $T$ is *not* stably infinite.

**All the other theories mentioned so far are stably infinite.**

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## Example: Theory of partial orders

$\Sigma$-theory $T_{\preceq}$

$\Sigma_{\preceq} : \ \{\preceq, \ =\}$

where $\preceq$ is a binary predicate.

Axioms

1. $\forall x.\ x \preceq x$ ($\preceq$ reflexivity)
2. $\forall x, y.\ x \preceq y \ \wedge \ y \preceq x \ \rightarrow \ x = y$ ($\preceq$ antisymmetry)
3. $\forall x, y, z.\ x \preceq y \ \wedge \ y \preceq z \ \rightarrow \ x \preceq z$ ($\preceq$ transitivity)

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We prove $T_{\preceq}$ is stably infinite.

Consider $T_{\preceq}$-satisfiable quantifier-free $\Sigma_{\preceq}$-formula $F$.
Consider arbitrary satisfying $T_{\preceq}$-interpretation $I : (D_I, \alpha_I)$,
where $\alpha_I$ maps $\preceq$ to $\leq_I$.

- Let $A$ be any infinite set disjoint from $D_I$
- Construct new interpretation $J : (D_J, \alpha_J)$
  - $D_J = D_I \cup A$
  - $\alpha_J = \{\preceq \mapsto \leq_J\}$, where for $a, b \in D_J$,

$$a \leq_J b \ \overset{\text{def}}{=} \ \begin{cases} a \leq_I b & \text{if } a, b \in D_I \\ a = b & \text{otherwise} \end{cases}$$

$J$ is $T_{\preceq}$-interpretation satisfying $F$ with infinite domain.
Hence, $T_{\preceq}$ is stably infinite.

## Example: Consider quantifier-free conjunctive $(\Sigma_E \cup \Sigma_{\mathbb{Z}})$-formula

$$F:\ 1 \leq x \ \wedge\ x \leq 2 \ \wedge\ f(x) \neq f(1) \ \wedge\ f(x) \neq f(2)\ .$$

The signatures of $T_E$ and $T_{\mathbb{Z}}$ only share $=$. Also, both theories are stably infinite. Hence, the NO combination of the decision procedures for $T_E$ and $T_{\mathbb{Z}}$ decides the $(T_E \cup T_{\mathbb{Z}})$-satisfiability of $F$.

Intuitively, $F$ is $(T_E \cup T_{\mathbb{Z}})$-unsatisfiable.
For the first two literals imply $x = 1 \ \vee\ x = 2$ so that
$f(x) = f(1) \ \vee\ f(x) = f(2)$.
Contradict last two literals.
Hence, $F$ is $(T_E \cup T_{\mathbb{Z}})$-unsatisfiable.

## N-O Overview

Phase 1: Variable Abstraction

- Given conjunction $\Gamma$ in theory $T_1 \cup T_2$.
- Convert to conjunction $\Gamma_1 \cup \Gamma_2$ s.t.
  - $\Gamma_i$ in theory $T_i$
  - $\Gamma_1 \cup \Gamma_2$ satisfiable iff $\Gamma$ satisfiable.

Phase 2: Check

- If there is some set $S$ of equalities and disequalities between the shared variables of $\Gamma_1$ and $\Gamma_2$
  shared$(\Gamma_1, \Gamma_2) =$ free$(\Gamma_1) \cap$ free$(\Gamma_2)$
  s.t. $S \cup \Gamma_i$ are $T_i$-satisfiable for all $i$,
  then $\Gamma$ is **satisfiable**.
- Otherwise, **unsatisfiable**.

## Nelson-Oppen Method: Overview

Consider quantifier-free conjunctive $(\Sigma_1 \cup \Sigma_2)$-formula $F$.

Two versions:

- nondeterministic — simple to present, but high complexity
- deterministic — efficient

Nelson-Oppen (N-O) method proceeds in two steps:

- Phase 1 (variable abstraction)
  — same for both versions
- Phase 2
  nondeterministic: guess equalities/disequalities and check
  deterministic: generate equalities/disequalities by equality propagation

## Phase 1: Variable abstraction

Given quantifier-free conjunctive $(\Sigma_1 \cup \Sigma_2)$-formula $F$.
Transform $F$ into two quantifier-free conjunctive formulae

$$\Sigma_1\text{-formula } F_1 \qquad \text{and} \qquad \Sigma_2\text{-formula } F_2$$

s.t. $F$ is $(T_1 \cup T_2)$-satisfiable iff $F_1 \ \wedge\ F_2$ is $(T_1 \cup T_2)$-satisfiable
$F_1$ and $F_2$ are linked via a set of shared variables.

For term $t$, let hd$(t)$ be the root symbol, e.g. hd$(f(x)) = f$.

### Generation of $F_1$ and $F_2$

For $i, j \in \{1, 2\}$ and $i \neq j$, repeat the transformations

(1) if function $f \in \Sigma_i$ and $\mathrm{hd}(t) \in \Sigma_j$,

$$F[f(t_1, \ldots, t, \ldots, t_n)] \quad \Rightarrow \quad F[f(t_1, \ldots, w, \ldots, t_n)] \wedge w = t$$

(2) if predicate $p \in \Sigma_i$ and $\mathrm{hd}(t) \in \Sigma_j$,

$$F[p(t_1, \ldots, t, \ldots, t_n)] \quad \Rightarrow \quad F[p(t_1, \ldots, w, \ldots, t_n)] \wedge w = t$$

(3) if $\mathrm{hd}(s) \in \Sigma_i$ and $\mathrm{hd}(t) \in \Sigma_j$,

$$F[s = t] \quad \Rightarrow \quad F[\top] \wedge w = s \wedge w = t$$

(4) if $\mathrm{hd}(s) \in \Sigma_i$ and $\mathrm{hd}(t) \in \Sigma_j$,

$$F[s \neq t] \quad \Rightarrow \quad F[w_1 \neq w_2] \wedge w_1 = s \wedge w_2 = t$$

where $w$, $w_1$, and $w_2$ are fresh variables.

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### Example: Consider $(\Sigma_E \cup \Sigma_{\mathbb{Z}})$-formula

$$F : \ 1 \leq x \ \wedge \ x \leq 2 \ \wedge \ f(x) \neq f(1) \ \wedge \ f(x) \neq f(2) \ .$$

According to transformation 1, since $f \in \Sigma_E$ and $1 \in \Sigma_{\mathbb{Z}}$, replace $f(1)$ by $f(w_1)$ and add $w_1 = 1$. Similarly, replace $f(2)$ by $f(w_2)$ and add $w_2 = 2$.

Now, the literals

$$\Gamma_{\mathbb{Z}} : \ \{1 \leq x, \ x \leq 2, \ w_1 = 1, \ w_2 = 2\}$$

are $T_{\mathbb{Z}}$-literals, while the literals

$$\Gamma_E : \ \{f(x) \neq f(w_1), \ f(x) \neq f(w_2)\}$$

are $T_E$-literals. Hence, construct the $\Sigma_{\mathbb{Z}}$-formula

$$F_1 : \ 1 \leq x \ \wedge \ x \leq 2 \ \wedge \ w_1 = 1 \ \wedge \ w_2 = 2$$

and the $\Sigma_E$-formula

$$F_2 : \ f(x) \neq f(w_1) \ \wedge \ f(x) \neq f(w_2) \ .$$

$F_1$ and $F_2$ share the variables $\{x, w_1, w_2\}$.
$F_1 \ \wedge \ F_2$ is $(T_E \cup T_{\mathbb{Z}})$-equisatisfiable to $F$.

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### Example: Consider $(\Sigma_E \cup \Sigma_{\mathbb{Z}})$-formula

$$F : \ f(x) = x+y \wedge x \leq y+z \wedge x+z \leq y \wedge y = 1 \wedge f(x) \neq f(2) \ .$$

In the first literal, $\mathrm{hd}(f(x)) = f \in \Sigma_=$ and $\mathrm{hd}(x + y) = + \in \Sigma_{\mathbb{Z}}$; thus, by (3), replace the literal with

$$w_1 = f(x) \ \wedge \ w_1 = x + y \ .$$

In the final literal, $f \in \Sigma_E$ but $2 \in \Sigma_{\mathbb{Z}}$, so by (1), replace it with

$$f(x) \neq f(w_2) \ \wedge \ w_2 = 2 \ .$$

Now, separating the literals results in two formulae:

$$F_1 : \ w_1 = x + y \ \wedge \ x \leq y + z \ \wedge \ x + z \leq y \ \wedge \ y = 1 \ \wedge \ w_2 = 2$$

is a $\Sigma_{\mathbb{Z}}$-formula, and

$$F_2 : \ w_1 = f(x) \ \wedge \ f(x) \neq f(w_2)$$

is a $\Sigma_E$-formula.
The conjunction $F_1 \ \wedge \ F_2$ is $(T_E \cup T_{\mathbb{Z}})$-equisatisfiable to $F$.

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## Nondeterministic Version

### Phase 2: Guess and Check

- Phase 1 separated $(\Sigma_1 \cup \Sigma_2)$-formula $F$ into two formulae:
  $\Sigma_1$-formula $F_1$ and $\Sigma_2$-formula $F_2$
- $F_1$ and $F_2$ are linked by a set of shared variables:
  $V = \mathrm{shared}(F_1, F_2) = \mathrm{free}(F_1) \cap \mathrm{free}(F_2)$
- Let $E$ be an equivalence relation over $V$.
- The arrangement $\alpha(V, E)$ of $V$ induced by $E$ is:

$$\alpha(V, E) : \quad \bigwedge_{u,v \ \in \ V. \ uEv} u = v \ \wedge \bigwedge_{u,v \ \in \ V. \ \neg(uEv)} u \neq v$$

Then,
the original formula $F$ is $(T_1 \cup T_2)$-satisfiable iff
there exists an equivalence relation $E$ of $V$ s.t.
(1) $F_1 \ \wedge \ \alpha(V, E)$ is $T_1$-satisfiable, and
(2) $F_2 \ \wedge \ \alpha(V, E)$ is $T_2$-satisfiable.
Otherwise, $F$ is $(T_1 \cup T_2)$-unsatisfiable.

Example: Consider $(\Sigma_E \cup \Sigma_\mathbb{Z})$-formula

$$F:\ 1 \le x \ \land\ x \le 2 \ \land\ f(x) \neq f(1) \ \land\ f(x) \neq f(2)$$

Phase 1 separates this formula into the $\Sigma_\mathbb{Z}$-formula

$$F_1:\ 1 \le x \ \land\ x \le 2 \ \land\ w_1 = 1 \ \land\ w_2 = 2$$

and the $\Sigma_E$-formula

$$F_2:\ f(x) \neq f(w_1) \ \land\ f(x) \neq f(w_2)$$

with

$$V = \text{shared}(F_1, F_2) = \{x, w_1, w_2\}$$

There are 5 equivalence relations to consider, which we list by stating the partitions:

1. $\{\{x, w_1, w_2\}\}$, *i.e.*, $x = w_1 = w_2$:
   $x = w_1$ and $f(x) \neq f(w_1)$ $\Rightarrow$ $F_2 \land \alpha(V,E)$ is $T_E$-unsatisfiable.
2. $\{\{x, w_1\}, \{w_2\}\}$, *i.e.*, $x = w_1$, $x \neq w_2$:
   $x = w_1$ and $f(x) \neq f(w_1)$ $\Rightarrow$ $F_2 \land \alpha(V,E)$ is $T_E$-unsatisfiable.
3. $\{\{x, w_2\}, \{w_1\}\}$, *i.e.*, $x = w_2$, $x \neq w_1$:
   $x = w_2$ and $f(x) \neq f(w_2)$ $\Rightarrow$ $F_2 \land \alpha(V,E)$ is $T_E$-unsatisfiable.
4. $\{\{x\}, \{w_1, w_2\}\}$, *i.e.*, $x \neq w_1$, $w_1 = w_2$:
   $w_1 = w_2$ and $w_1 = 1 \ \land\ w_2 = 2$
   $\Rightarrow$ $F_1 \land \alpha(V,E)$ is $T_\mathbb{Z}$-unsatisfiable.
5. $\{\{x\}, \{w_1\}, \{w_2\}\}$, *i.e.*, $x \neq w_1$, $x \neq w_2$, $w_1 \neq w_2$:
   $x \neq w_1 \ \land\ x \neq w_2$ and $x = w_1 = 1 \ \lor\ x = w_2 = 2$
   (since $1 \le x \le 2$ implies that $x = 1 \ \lor\ x = 2$ in $T_\mathbb{Z}$)
   $\Rightarrow$ $F_1 \land \alpha(V,E)$ is $T_\mathbb{Z}$-unsatisfiable.

Hence, $F$ is $(T_E \cup T_\mathbb{Z})$-unsatisfiable.

Example: Consider the $(\Sigma_{\text{cons}} \cup \Sigma_\mathbb{Z})$-formula

$$F:\ \text{car}(x) + \text{car}(y) = z \ \land\ \text{cons}(x, z) \neq \text{cons}(y, z)\ .$$

After two applications of (1), Phase 1 separates $F$ into the $\Sigma_{\text{cons}}$-formula

$$F_1:\ w_1 = \text{car}(x) \ \land\ w_2 = \text{car}(y) \ \land\ \text{cons}(x, z) \neq \text{cons}(y, z)$$

and the $\Sigma_\mathbb{Z}$-formula

$$F_2:\ w_1 + w_2 = z\ ,$$

with

$$V = \text{shared}(F_1, F_2) = \{z, w_1, w_2\}\ .$$

Consider the equivalence relation $E$ given by the partition

$$\{\{z\}, \{w_1\}, \{w_2\}\}\ .$$

The arrangement

$$\alpha(V,E):\ z \neq w_1 \ \land\ z \neq w_2 \ \land\ w_1 \neq w_2$$

satisfies both $F_1$ and $F_2$: $F_1 \land \alpha(V,E)$ is $T_{\text{cons}}$-satisfiable, and $F_2 \land \alpha(V,E)$ is $T_\mathbb{Z}$-satisfiable.
Hence, $F$ is $(T_{\text{cons}} \cup T_\mathbb{Z})$-satisfiable.

## Practical Efficiency

Phase 2 was formulated as "guess and check":
First, guess an equivalence relation $E$,
then check the induced arrangement.

The number of equivalence relations grows super-exponentially with the # of shared variables. It is given by Bell numbers.
e.g., 12 shared variables $\Rightarrow$ over four million equivalence relations.

Solution: Deterministic Version

## Deterministic Version

Phase 1 as before

Phase 2 asks the decision procedures $P_1$ and $P_2$ to propagate new equalities.

Example 1:

Real linear arithmetic $T_{\mathbb{R}}$ $\boxed{P_{\mathbb{R}}}$

Theory of equality $T_E$ $\boxed{P_E}$

$$F: \quad f(f(x)-f(y)) \neq f(z) \ \wedge \ x \leq y \ \wedge \ y+z \leq x \ \wedge \ 0 \leq z$$

$(T_{\mathbb{R}} \cup T_E)$-unsatisfiable

Intuitively,
last 3 conjuncts $\Rightarrow$ $x = y \ \wedge \ z = 0$
contradicts 1st conjunct

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### Phase 1: Variable Abstraction

$$F: \ f(f(x)-f(y)) \neq f(z) \ \wedge \ x \leq y \ \wedge \ y+z \leq x \ \wedge \ 0 \leq z$$

$$f(x) \ \Rightarrow \ u \qquad f(y) \ \Rightarrow \ v \qquad u-v \ \Rightarrow \ w$$

$\Gamma_E: \quad \{f(w) \neq f(z), \ u = f(x), \ v = f(y)\}$ $\quad \ldots$ $T_E$-formula

$\Gamma_{\mathbb{R}}: \quad \{x \leq y, \ y+z \leq x, \ 0 \leq z, \ w = u - v\}$ $\quad \ldots$ $T_{\mathbb{R}}$-formula

$$\text{shared}(\Gamma_{\mathbb{R}}, \Gamma_E) = \{x, y, z, u, v, w\}$$

Nondeterministic version — over 200 $E$s!
Let's try the deterministic version.

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### Phase 2: Equality Propagation

$\boxed{P_{\mathbb{R}}}$ $\qquad\qquad$ $\boxed{P_E}$

$s_0: \langle \Gamma_{\mathbb{R}}, \Gamma_E, \{\} \rangle$

$\Gamma_{\mathbb{R}} \models x = y$

$s_1: \langle \Gamma_{\mathbb{R}}, \Gamma_E, \{x = y\} \rangle$

$\Gamma_E \cup \{x = y\} \models u = v$

$s_2: \langle \Gamma_{\mathbb{R}}, \Gamma_E, \{x = y, u = v\} \rangle$

$\Gamma_{\mathbb{R}} \cup \{u = v\} \models z = w$

$s_3: \langle \Gamma_{\mathbb{R}}, \Gamma_E, \{x = y, u = v, z = w\} \rangle$

$\Gamma_E \cup \{z = w\} \models \text{false}$

$s_4: \text{false}$

Contradiction. Thus, $F$ is $(T_{\mathbb{R}} \cup T_E)$-unsatisfiable.
If there were no contradiction, $F$ would be $(T_{\mathbb{R}} \cup T_E)$-satisfiable.

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### Convex Theories

**Claim**:
Equality propagation is a decision procedure for convex theories.

**Def.** A $\Sigma$-theory $T$ is *convex* iff
for every quantifier-free conjunction $\Sigma$-formula $F$
and for every disjunction $\bigvee_{i=1}^{n} (u_i = v_i)$

if $F \ \models \ \bigvee_{i=1}^{n} (u_i = v_i)$

then $F \ \models \ u_i = v_i$, for some $i \in \{1, \ldots, n\}$

## Convex Theories

- $T_E$, $T_{\mathbb{R}}$, $T_{\mathbb{Q}}$, $T_{\text{cons}}$ are convex
- $T_{\mathbb{Z}}$, $T_{\mathsf{A}}$ are not convex

Example: $T_{\mathbb{Z}}$ is not convex

Consider quantifier-free conjunctive

$$F: \quad 1 \leq z \ \wedge \ z \leq 2 \ \wedge \ u = 1 \ \wedge \ v = 2$$

Then

$$F \ \models \ z = u \vee z = v$$

but

$$F \ \not\models \ z = u$$
$$F \ \not\models \ z = v$$

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## Example:

The theory of arrays $T_{\mathsf{A}}$ is not convex.
Consider the quantifier-free conjunctive $\Sigma_{\mathsf{A}}$-formula

$$F: \ a\langle i \triangleleft v\rangle[j] = v \ .$$

Then

$$F \ \Rightarrow \ i = j \ \vee \ a[j] = v \ ,$$

but

$$F \not\Rightarrow i = j$$
$$F \not\Rightarrow a[j] = v \ .$$

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## What if $T$ is Not Convex?

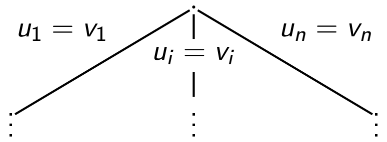
Case split when:

$$\Gamma \models \bigvee_{i=1}^{n} (u_i = v_i)$$

but

$$\Gamma \not\models u_i = v_i \qquad\qquad \text{for all } i = 1, \ldots, n$$

- For each $i = 1, \ldots, n$, construct a branch on which $u_i = v_i$ is assumed.
- If all branches are contradictory, then **unsatisfiable**. Otherwise, **satisfiable**.



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## Example 2: Non-Convex Theory

$T_{\mathbb{Z}}$ not convex! $\boxed{P_{\mathbb{Z}}}$

$T_E$ convex $\boxed{P_E}$

$$\Gamma: \left\{ \begin{array}{ll} 1 \leq x, & x \leq 2, \\ f(x) \neq f(1), & f(x) \neq f(2) \end{array} \right\} \quad \text{in } T_{\mathbb{Z}} \cup T_E$$

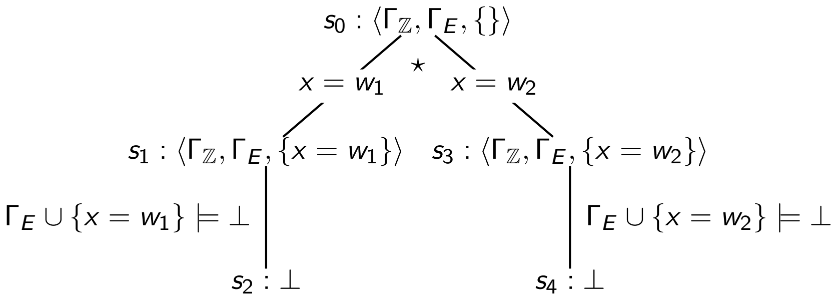
- Replace $f(1)$ by $f(w_1)$, and add $w_1 = 1$.
- Replace $f(2)$ by $f(w_2)$, and add $w_2 = 2$.

Result:

$$\Gamma_{\mathbb{Z}} = \left\{ \begin{array}{l} 1 \leq x, \\ x \leq 2, \\ w_1 = 1, \\ w_2 = 2 \end{array} \right\} \quad \text{and} \quad \Gamma_E = \left\{ \begin{array}{l} f(x) \neq f(w_1), \\ f(x) \neq f(w_2) \end{array} \right\}$$

$\text{shared}(\Gamma_{\mathbb{Z}}, \Gamma_E) = \{x, w_1, w_2\}$

## Example 2: Non-Convex Theory



$s_0 : \langle \Gamma_{\mathbb{Z}}, \Gamma_E, \{\} \rangle$

$\star$

$x = w_1$

$x = w_2$

$s_1 : \langle \Gamma_{\mathbb{Z}}, \Gamma_E, \{x = w_1\} \rangle$

$s_3 : \langle \Gamma_{\mathbb{Z}}, \Gamma_E, \{x = w_2\} \rangle$

$\Gamma_E \cup \{x = w_1\} \models \bot$

$\Gamma_E \cup \{x = w_2\} \models \bot$

$s_2 : \bot$

$s_4 : \bot$

$\star$: $\Gamma_{\mathbb{Z}} \models x = w_1 \ \vee \ x = w_2$

All leaves are labeled with $\bot \Rightarrow \Gamma$ is $(T_{\mathbb{Z}} \cup T_E)$-unsatisfiable.

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## Example 3: Non-Convex Theory

$$\Gamma : \left\{ \begin{array}{c} 1 \leq x, \quad x \leq 3, \\ f(x) \neq f(1), \ f(x) \neq f(3), \ f(1) \neq f(2) \end{array} \right\} \quad \text{in } T_{\mathbb{Z}} \cup T_E$$

- Replace $f(1)$ by $f(w_1)$, and add $w_1 = 1$.
- Replace $f(2)$ by $f(w_2)$, and add $w_2 = 2$.
- Replace $f(3)$ by $f(w_3)$, and add $w_3 = 3$.

Result:

$$\Gamma_{\mathbb{Z}} = \left\{ \begin{array}{l} 1 \leq x, \\ x \leq 3, \\ w_1 = 1, \\ w_2 = 2, \\ w_3 = 3 \end{array} \right\} \quad \text{and} \quad \Gamma_E = \left\{ \begin{array}{l} f(x) \neq f(w_1), \\ f(x) \neq f(w_3), \\ f(w_1) \neq f(w_2) \end{array} \right\}$$

$\text{shared}(\Gamma_{\mathbb{Z}}, \Gamma_E) = \{x, w_1, w_2, w_3\}$

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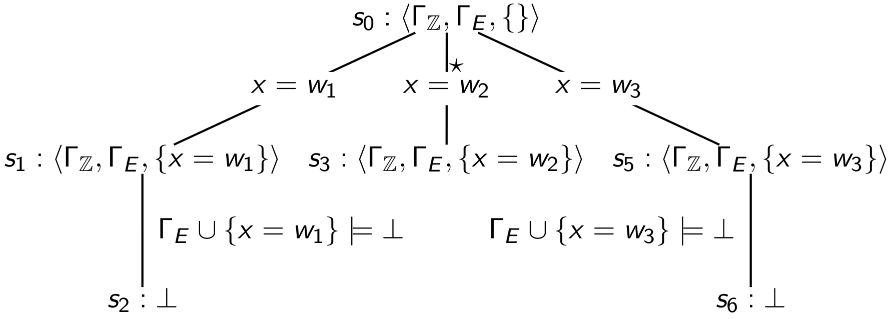
## Example 3: Non-Convex Theory



$s_0 : \langle \Gamma_{\mathbb{Z}}, \Gamma_E, \{\} \rangle$

$\star$

$x = w_1$

$x = w_2$

$x = w_3$

$s_1 : \langle \Gamma_{\mathbb{Z}}, \Gamma_E, \{x = w_1\} \rangle$

$s_3 : \langle \Gamma_{\mathbb{Z}}, \Gamma_E, \{x = w_2\} \rangle$

$s_5 : \langle \Gamma_{\mathbb{Z}}, \Gamma_E, \{x = w_3\} \rangle$

$\Gamma_E \cup \{x = w_1\} \models \bot$

$\Gamma_E \cup \{x = w_3\} \models \bot$

$s_2 : \bot$

$s_6 : \bot$

$\star$: $\Gamma_{\mathbb{Z}} \models x = w_1 \ \vee \ x = w_2 \ \vee \ x = w_3$

No more equations on middle leaf $\Rightarrow \Gamma$ is $(T_{\mathbb{Z}} \cup T_E)$-satisfiable.