

2. First-Order Logic (FOL)

First-Order Logic (FOL)

Also called Predicate Logic or Predicate Calculus

FOL Syntax

variables x, y, z, \dots

constants a, b, c, \dots

functions f, g, h, \dots

terms variables, constants or
n-ary function applied to n terms as arguments
 $a, x, f(a), g(x, b), f(g(x, g(b)))$

predicates p, q, r, \dots

atom \top, \perp , or an n-ary predicate applied to n terms

literal atom or its negation
 $p(f(x), g(x, f(x))), \quad \neg p(f(x), g(x, f(x)))$

Note: 0-ary functions: constant
0-ary predicates: P, Q, R, \dots

quantifiers

existential quantifier $\exists x.F[x]$

“there exists an x such that $F[x]$ ”

universal quantifier $\forall x.F[x]$

“for all x , $F[x]$ ”

FOL formula literal, application of logical connectives

(\neg , \vee , \wedge , \rightarrow , \leftrightarrow) to formulae,

or application of a quantifier to a formula

Example: FOL formula

$$\forall x. \underbrace{p(f(x), x) \rightarrow (\exists y. \underbrace{p(f(g(x, y)), g(x, y))}_{G}) \wedge q(x, f(x))}_{F}$$

The scope of $\forall x$ is F .

The scope of $\exists y$ is G .

The formula reads:

“for all x ,

if $p(f(x), x)$

then there exists a y such that

$p(f(g(x, y)), g(x, y))$ and $q(x, f(x))$ ”

Translations of English Sentences into FOL

- ▶ The length of one side of a triangle is less than the sum of the lengths of the other two sides

$$\forall x, y, z. \text{triangle}(x, y, z) \rightarrow \text{length}(x) < \text{length}(y) + \text{length}(z)$$

- ▶ Fermat's Last Theorem.

$$\forall n. \text{integer}(n) \wedge n > 2$$

$$\rightarrow \forall x, y, z.$$

$$\text{integer}(x) \wedge \text{integer}(y) \wedge \text{integer}(z)$$

$$\wedge x > 0 \wedge y > 0 \wedge z > 0$$

$$\rightarrow x^n + y^n \neq z^n$$

FOL Semantics

An interpretation $I : (D_I, \alpha_I)$ consists of:

- ▶ Domain D_I
non-empty set of values or objects
cardinality $|D_I|$ finite (eg, 52 cards),
countably infinite (eg, integers), or
uncountably infinite (eg, reals)
- ▶ Assignment α_I
 - ▶ each variable x assigned value $x_I \in D_I$
 - ▶ each n-ary function f assigned

$$f_I : D_I^n \rightarrow D_I$$

In particular, each constant a (0-ary function) assigned value $a_I \in D_I$

- ▶ each n-ary predicate p assigned

$$p_I : D_I^n \rightarrow \{\underline{\text{true}}, \underline{\text{false}}\}$$

In particular, each propositional variable P (0-ary predicate) assigned truth value (true, false)

Example:

$$F : p(f(x, y), z) \rightarrow p(y, g(z, x))$$

Interpretation $I : (D_I, \alpha_I)$

$$D_I = \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\} \quad \text{integers}$$

$$\alpha_I : \{f \mapsto +, g \mapsto -, p \mapsto >\}$$

Therefore, we can write

$$F_I : x + y > z \rightarrow y > z - x$$

(This is the way we'll write it in the future!)

Also

$$\alpha_I : \{x \mapsto 13, y \mapsto 42, z \mapsto 1\}$$

Thus

$$F_I : 13 + 42 > 1 \rightarrow 42 > 1 - 13$$

Compute the truth value of F under I

1. $I \models x + y > z$ since $13 + 42 > 1$
2. $I \models y > z - x$ since $42 > 1 - 13$
3. $I \models F$ by 1, 2, and \rightarrow

F is true under I

Semantics: Quantifiers

x variable.

x -variant of interpretation I is an interpretation $J : (D_J, \alpha_J)$ such that

- ▶ $D_I = D_J$
- ▶ $\alpha_I[y] = \alpha_J[y]$ for all symbols y , except possibly x

That is, I and J agree on everything except possibly the value of x

Denote $J : I \triangleleft \{x \mapsto v\}$ the x -variant of I in which $\alpha_J[x] = v$ for some $v \in D_I$. Then

- ▶ $I \models \forall x. F$ iff for all $v \in D_I$, $I \triangleleft \{x \mapsto v\} \models F$
- ▶ $I \models \exists x. F$ iff there exists $v \in D_I$ s.t. $I \triangleleft \{x \mapsto v\} \models F$

Example

For \mathbb{Q} , the set of rational numbers, consider

$$F_I : \forall x. \exists y. 2 \times y = x$$

Compute the value of F_I (F under I):

Let

$$J_1 : I \triangleleft \{x \mapsto v\}$$

x-variant of I

$$J_2 : J_1 \triangleleft \{y \mapsto \frac{v}{2}\}$$

y-variant of J_1

for $v \in \mathbb{Q}$.

Then

1. $J_2 \models 2 \times y = x$ since $2 \times \frac{v}{2} = v$
2. $J_1 \models \exists y. 2 \times y = x$
3. $I \models \forall x. \exists y. 2 \times y = x$ since $v \in \mathbb{Q}$ is arbitrary

Satisfiability and Validity

F is satisfiable iff there exists I s.t. $I \models F$

F is valid iff for all I , $I \models F$

F is valid iff $\neg F$ is unsatisfiable

Example: $F : (\forall x. p(x)) \leftrightarrow (\neg \exists x. \neg p(x))$ valid?

Suppose not. Then there is I s.t.

$$0. \quad I \not\models (\forall x. p(x)) \leftrightarrow (\neg \exists x. \neg p(x))$$

First case

1. $I \models \forall x. p(x)$ assumption
2. $I \not\models \neg \exists x. \neg p(x)$ assumption
3. $I \models \exists x. \neg p(x)$ 2 and \neg
4. $I \triangleleft \{x \mapsto v\} \models \neg p(x)$ 3 and \exists , for some $v \in D_I$
5. $I \triangleleft \{x \mapsto v\} \models p(x)$ 1 and \forall

4 and 5 are contradictory.

Second case

- | | | | | |
|----|-----------------------------------|---------------|-----------------------------|--|
| 1. | I | $\not\models$ | $\forall x. p(x)$ | assumption |
| 2. | I | \models | $\neg \exists x. \neg p(x)$ | assumption |
| 3. | $I \triangleleft \{x \mapsto v\}$ | $\not\models$ | $p(x)$ | 1 and \forall , for some $v \in D_I$ |
| 4. | I | $\not\models$ | $\exists x. \neg p(x)$ | 2 and \neg |
| 5. | $I \triangleleft \{x \mapsto v\}$ | $\not\models$ | $\neg p(x)$ | 4 and \exists |
| 6. | $I \triangleleft \{x \mapsto v\}$ | \models | $p(x)$ | 5 and \neg |

3 and 6 are contradictory.

Both cases end in contradictions for arbitrary $I \Rightarrow F$ is valid.

Example: Prove

$F : p(a) \rightarrow \exists x. p(x)$ is valid.

Assume otherwise.

- | | | | | |
|----|---|---------------|-------------------|---------------------|
| 1. | I | $\not\models$ | F | assumption |
| 2. | I | \models | $p(a)$ | 1 and \rightarrow |
| 3. | I | $\not\models$ | $\exists x. p(x)$ | 1 and \rightarrow |
| 4. | $I \triangleleft \{x \mapsto \alpha_I[a]\}$ | $\not\models$ | $p(x)$ | 3 and \exists |

2 and 4 are contradictory. Thus, F is valid.

Example: Show

$F : (\forall x. p(x, x)) \rightarrow (\exists x. \forall y. p(x, y))$ is invalid.

Find interpretation I such that

$$I \models \neg[(\forall x. p(x, x)) \rightarrow (\exists x. \forall y. p(x, y))]$$

i.e.

$$I \models (\forall x. p(x, x)) \wedge \neg(\exists x. \forall y. p(x, y))$$

Choose $D_I = \{0, 1\}$

$p_I = \{(0, 0), (1, 1)\}$ i.e. $p_I(0, 0)$ and $p_I(1, 1)$ are true
 $p_I(1, 0)$ and $p_I(0, 1)$ are false

I falsifying interpretation $\Rightarrow F$ is invalid.

Safe Substitution $F\sigma$

Example:

$$F : (\forall x. \overbrace{p(x, y)}^{\text{scope of } \forall x}) \rightarrow q(f(y), x)$$

bound by $\forall x$ ↗ ↖ free free ↗ ↖ free

$$\text{free}(F) = \{x, y\}$$

substitution

$$\sigma : \{x \mapsto g(x), y \mapsto f(x), q(f(y), x) \mapsto \exists x. h(x, y)\}$$

$F\sigma$?

1. Rename

$$F' : \forall x'. p(x', y) \rightarrow q(f(y), x)$$

↑ ↑

where x' is a fresh variable

2. $F'\sigma : \forall x'. p(x', f(x)) \rightarrow \exists x. h(x, y)$

Rename x by x' :

replace x in $\forall x$ by x' and all free x in the scope of $\forall x$ by x' .

$$\forall x. G[x] \quad \Leftrightarrow \quad \forall x'. G[x']$$

Same for $\exists x$

$$\exists x. G[x] \quad \Leftrightarrow \quad \exists x'. G[x']$$

where x' is a fresh variable

Proposition (Substitution of Equivalent Formulae)

$$\sigma : \{F_1 \mapsto G_1, \dots, F_n \mapsto G_n\}$$

s.t. for each i , $F_i \Leftrightarrow G_i$

If $F\sigma$ a safe substitution, then $F \Leftrightarrow F\sigma$

Formula Schema

Formula

$$(\forall x. p(x)) \leftrightarrow (\neg \exists x. \neg p(x))$$

Formula Schema

$$H_1 : (\forall x. F) \leftrightarrow (\neg \exists x. \neg F)$$

↑ place holder

Formula Schema (with side condition)

$$H_2 : (\forall x. F) \leftrightarrow F \quad \text{provided } x \notin \text{free}(F)$$

Valid Formula Schema

H is valid iff valid for any FOL formula F_i obeying the side conditions

Example: H_1 and H_2 are valid.

Substitution σ of H

$$\sigma : \{F_1 \mapsto \quad, \dots, F_n \mapsto \quad\}$$

mapping place holders F_i of H to FOL formulae,
(obeying the side conditions of H)

Proposition (Formula Schema)

If H is valid formula schema and
 σ is a substitution obeying H 's side conditions
then $H\sigma$ is also valid.

Example:

$H : (\forall x. F) \leftrightarrow F$ provided $x \notin \text{free}(F)$ is valid

$\sigma : \{F \mapsto p(y)\}$ obeys the side condition

Therefore $H\sigma : \forall x. p(y) \leftrightarrow p(y)$ is valid

Proving Validity of Formula Schema

Example: Prove validity of

$$H : (\forall x. F) \leftrightarrow F \quad \text{provided } x \notin \text{free}(F)$$

Proof by contradiction. Consider the two directions of \leftrightarrow .

First case:

1. $I \models \forall x. F$ assumption
2. $I \not\models F$ assumption
3. $I \models F$ 1, \forall , since $x \notin \text{free}(F)$
4. $I \models \perp$ 2, 3

Second Case:

1. $I \not\models \forall x. F$ assumption
2. $I \models F$ assumption
3. $I \models \exists x. \neg F$ 1 and \neg
4. $I \models \neg F$ 3, \exists , since $x \notin \text{free}(F)$
5. $I \models \perp$ 2, 4

Hence, H is a valid formula schema.

Normal Forms

1. Negation Normal Forms (NNF)

Augment the equivalence with (left-to-right)

$$\neg\forall x. F[x] \Leftrightarrow \exists x. \neg F[x]$$

$$\neg\exists x. F[x] \Leftrightarrow \forall x. \neg F[x]$$

Example

$$G : \forall x. (\exists y. p(x, y) \wedge p(x, z)) \rightarrow \exists w. p(x, w) .$$

1. $\forall x. (\exists y. p(x, y) \wedge p(x, z)) \rightarrow \exists w. p(x, w)$

2. $\forall x. \neg(\exists y. p(x, y) \wedge p(x, z)) \vee \exists w. p(x, w)$

$$F_1 \rightarrow F_2 \Leftrightarrow \neg F_1 \vee F_2$$

3. $\forall x. (\forall y. \neg(p(x, y) \wedge p(x, z))) \vee \exists w. p(x, w)$

$$\neg\exists x. F[x] \Leftrightarrow \forall x. \neg F[x]$$

4. $\forall x. (\forall y. \neg p(x, y) \vee \neg p(x, z)) \vee \exists w. p(x, w)$

2. Prenex Normal Form (PNF)

All quantifiers appear at the beginning of the formula

$$Q_1 x_1 \cdots Q_n x_n. F[x_1, \dots, x_n]$$

where $Q_i \in \{\forall, \exists\}$ and F is quantifier-free.

Every FOL formula F can be transformed to formula F' in PNF
s.t. $F' \Leftrightarrow F$.

Example: Find equivalent PNF of

$$F : \forall x. \neg(\exists y. p(x, y) \wedge p(x, z)) \vee \exists y. p(x, y)$$

↑ to the end of the formula

1. Write F in NNF

$$F_1 : \forall x. (\forall y. \neg p(x, y) \vee \neg p(x, z)) \vee \exists y. p(x, y)$$

2. Rename quantified variables to fresh names

$$F_2 : \forall x. (\forall y. \neg p(x, y) \vee \neg p(x, z)) \vee \exists w. p(x, w)$$

↑ in the scope of $\forall x$

3. Remove all quantifiers to produce quantifier-free formula

$$F_3 : \neg p(x, y) \vee \neg p(x, z) \vee p(x, w)$$

4. Add the quantifiers before F_3

$$F_4 : \forall x. \forall y. \exists w. \neg p(x, y) \vee \neg p(x, z) \vee p(x, w)$$

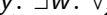
Alternately,

$$F'_4 : \forall x. \exists w. \forall y. \neg p(x, y) \vee \neg p(x, z) \vee p(x, w)$$

Note: In F_2 , $\forall y$ is in the scope of $\forall x$, therefore the order of quantifiers must be $\dots \forall x \dots \forall y \dots$

$F_4 \Leftrightarrow F \text{ and } F'_4 \Leftrightarrow F$

Note: However $G \not\Leftrightarrow F$

$$G : \forall y. \exists w. \forall x. \neg p(x, y) \vee \neg p(x, z) \vee p(x, w)$$


Decidability of FOL

- ▶ FOL is undecidable (Turing & Church)
There does not exist an algorithm for deciding if a FOL formula F is valid, i.e. always halt and says “yes” if F is valid or say “no” if F is invalid.
- ▶ FOL is semi-decidable
There is a procedure that always halts and says “yes” if F is valid, but may not halt if F is invalid.

On the other hand,

- ▶ PL is decidable
There does exist an algorithm for deciding if a PL formula F is valid, e.g. the truth-table procedure.

Similarly for satisfiability

Semantic Argument Proof

To show FOL formula F is valid, assume $I \not\models F$ and derive a contradiction $I \models \perp$ in all branches

- ▶ Soundness

If every branch of a semantic argument proof reach $I \models \perp$, then F is valid

- ▶ Completeness

Each valid formula F has a semantic argument proof in which every branch reach $I \models \perp$