4. Induction

Induction

- Stepwise induction (for T_{PA}, T_{cons})
- Complete induction (for T_{PA}, T_{cons})
  Theoretically equivalent in power to stepwise induction,
  but sometimes produces more concise proof
- Well-founded induction
  Generalized complete induction
- Structural induction
  Over logical formulae

Stepwise Induction (Peano Arithmetic T_{PA})

Axiom schema (induction)

$F[0]\ \wedge$ . . . base case
$(\forall n.\ F[n] \ \rightarrow\ F[n+1])$ . . . inductive step
$\rightarrow\ \forall x.\ F[x]$ . . . conclusion

for Σ_{PA}-formulae $F[x]$ with one free variable x.

To prove $\forall x.\ F[x]$, i.e.,
$F[x]$ is T_{PA}-valid for all $x \in \mathbb{N}$,
it suffices to show

- base case: prove $F[0]$ is T_{PA}-valid.
- inductive step: For arbitrary $n \in \mathbb{N}$,
  assume inductive hypothesis, i.e.,
  $F[n]$ is T_{PA}-valid,
  then prove the conclusion
  $F[n+1]$ is T_{PA}-valid.

Example:

Theory T^{+}_{PA} obtained from T_{PA} by adding the axioms:

- $\forall x.\ x^0 = 1$ (E0)
- $\forall x, y.\ x^{y+1} = x^y \cdot x$ (E1)
- $\forall x, z.\ exp_3(x, 0, z) = z$ (P0)
- $\forall x, y, z.\ exp_3(x, y+1, z) = exp_3(x, y, x \cdot z)$ (P1)

Prove that

$$\boxed{\forall x, y.\ exp_3(x, y, 1) = x^y}$$

is T^{+}_{PA}-valid.

First attempt:

$$\forall y\ [\underbrace{\forall x.\ exp_3(x,y,1) = x^y}_{F[y]}]$$

We chose induction on y. Why?

Base case:

$F[0]:\ \forall x.\ exp_3(x,0,1) = x^0$

OK since $exp_3(x,0,1) = 1$ (P0) and $x^0 = 1$ (E0).

Inductive step: Failure.

For arbitrary $n \in \mathbb{N}$, we cannot deduce

$F[n+1] : \forall x.\ exp_3(x,n+1,1) = x^{n+1}$

from the inductive hypothesis

$F[n] : \forall x.\ exp_3(x,n,1) = x^n$

Second attempt: Strengthening

Strengthened property

$$\boxed{\forall x,y,z.\ exp_3(x,y,z) = x^y \cdot z}$$

Implies the desired property (choose $z = 1$)

$\forall x,y.\ exp_3(x,y,1) = x^y$

Again, induction on y

$$\forall y\ [\underbrace{\forall x,z.\ exp_3(x,y,z) = x^y \cdot z}_{F[y]}]$$

Base case:

$F[0]:\ \forall x,z.\ exp_3(x,0,z) = x^0 \cdot z$

OK since $exp_3(x,0,z) = z$ (P0) and $x^0 = 1$ (E0).

Inductive step: For arbitrary $n \in \mathbb{N}$

Assume inductive hypothesis

$$F[n] : \forall x,z.\ exp_3(x,n,z) = x^n \cdot z \qquad \text{(IH)}$$

prove

$$F[n+1] : \forall x,\underset{\uparrow}{z'}.\ exp_3(x,n+1,z') = x^{n+1} \cdot z'$$

$$\begin{aligned} exp_3(x,n+1,z') &= exp_3(x,n,x\cdot z') && \text{(P1)} \\ &= x^n \cdot (x \cdot z') && \text{IH } F[n], z \mapsto x \cdot z' \\ &= x^{n+1} \cdot z' && \text{(E1)} \end{aligned}$$

Stepwise Induction (Lists T_{cons})

Axiom schema (induction)

$(\forall$ atom $u.\ F[u] \wedge$ . . . base case

$(\forall u,v.\ F[v] \ \rightarrow\ F[\text{cons}(u,v)])$ . . . inductive step

$\rightarrow\ \forall x.\ F[x]$ . . . conclusion

for Σ_{cons}-formulae $F[x]$ with one free variable x.

To prove $\forall x.\ F[x]$, i.e.,

$F[x]$ is T_{cons}-valid for all lists x,

it suffices to show

- base case: prove $F[u]$ is T_{cons}-valid for arbitrary atom u.
- inductive step: For arbitrary list v,
  assume inductive hypothesis, i.e.,
  $F[v]$ is T_{cons}-valid,
  then prove the conclusion
  $F[\text{cons}(u,v)]$ is T_{cons}-valid for arbitrary atom u.

Example

Theory T^{+}_{cons} obtained from T_{cons} by adding the axioms for concatenating two lists, reverse a list, and decide if a list is flat (i.e., $\mathit{flat}(x)$ is $\top$ iff every element of list x is an atom).

- $\forall$ atom u. $\forall v$. $\mathit{concat}(u, v) = \text{cons}(u, v)$ (C0)
- $\forall u, v, x$. $\mathit{concat}(\text{cons}(u, v), x) = \text{cons}(u, \mathit{concat}(v, x))$ (C1)
- $\forall$ atom u. $\mathit{rvs}(u) = u$ (R0)
- $\forall x, y$. $\mathit{rvs}(\mathit{concat}(x, y)) = \mathit{concat}(\mathit{rvs}(y), \mathit{rvs}(x))$ (R1)
- $\forall$ atom u. $\mathit{flat}(u)$ (F0)
- $\forall u, v$. $\mathit{flat}(\text{cons}(u, v)) \leftrightarrow \text{atom}(u) \wedge \mathit{flat}(v)$ (F1)

Prove

$$\boxed{\forall x.\ \mathit{flat}(x) \rightarrow \mathit{rvs}(\mathit{rvs}(x)) = x}$$

is T^{+}_{cons}-valid.

Base case: For arbitrary atom u,

$$F[u] : \mathit{flat}(u) \rightarrow \mathit{rvs}(\mathit{rvs}(u)) = u$$

by R0.

Inductive step: For arbitrary lists u, v,

assume the inductive hypothesis

$$F[v] :\ \mathit{flat}(v) \rightarrow \mathit{rvs}(\mathit{rvs}(v)) = v \qquad \text{(IH)}$$

Prove

$$F[\text{cons}(u, v)] :\ \mathit{flat}(\text{cons}(u, v)) \rightarrow \mathit{rvs}(\mathit{rvs}(\text{cons}(u, v))) = \text{cons}(u, v) \qquad (*)$$

Case $\neg\text{atom}(u)$

$$\mathit{flat}(\text{cons}(u, v)) \Leftrightarrow \text{atom}(u) \wedge \mathit{flat}(v) \Leftrightarrow \bot$$

by (F1). $(*)$ holds since its antecedent is $\bot$.

Case $\text{atom}(u)$

$$\mathit{flat}(\text{cons}(u, v)) \Leftrightarrow \text{atom}(u) \wedge \mathit{flat}(v) \Leftrightarrow \mathit{flat}(v)$$

by (F1).

$$\mathit{rvs}(\mathit{rvs}(\text{cons}(u, v))) = \cdots = \text{cons}(u, v).$$

Complete Induction (Peano Arithmetic T_{PA})

Axiom schema (complete induction)

$$(\forall n.\ (\forall n'.\ n' < n \rightarrow F[n']) \rightarrow F[n]) \qquad \ldots \text{ inductive step}$$
$$\rightarrow \forall x.\ F[x] \qquad \ldots \text{ conclusion}$$

for Σ_{PA}-formulae $F[x]$ with one free variable x.

To prove $\forall x.\ F[x]$, i.e.,

$F[x]$ is T_{PA}-valid for all $x \in \mathbb{N}$,

it suffices to show

- inductive step: For arbitrary $n \in \mathbb{N}$,
  assume inductive hypothesis, i.e.,
  $F[n']$ is T_{PA}-valid for every $n' \in \mathbb{N}$ such that $n' < n$,
  then prove
  $F[n]$ is T_{PA}-valid.

Is base case missing?

No. Base case is implicit in the structure of complete induction.

Note:

- Complete induction is theoretically equivalent in power to stepwise induction.
- Complete induction sometimes yields more concise proofs.

Example: Integer division $\quad \mathit{quot}(5, 3) = 1$ and $\mathit{rem}(5, 3) = 2$

Theory T^{*}_{PA} obtained from T_{PA} by adding the axioms:

- $\forall x, y.\ x < y \rightarrow \mathit{quot}(x, y) = 0$ (Q0)
- $\forall x, y.\ y > 0 \rightarrow \mathit{quot}(x + y, y) = \mathit{quot}(x, y) + 1$ (Q1)
- $\forall x, y.\ x < y \rightarrow \mathit{rem}(x, y) = x$ (R0)
- $\forall x, y.\ y > 0 \rightarrow \mathit{rem}(x + y, y) = \mathit{rem}(x, y)$ (R1)

Prove

(1) $\forall x, y.\ y > 0 \rightarrow \mathit{rem}(x, y) < y$
(2) $\forall x, y.\ y > 0 \rightarrow x = y \cdot \mathit{quot}(x, y) + \mathit{rem}(x, y)$

Best proved by complete induction.

Proof of (1)

$$\forall x.\ \underbrace{\forall y.\ y > 0 \ \rightarrow\ rem(x, y) < y}_{F[x]}$$

Consider an arbitrary natural number x.
Assume the inductive hypothesis

$$\forall x'.\ x' < x \ \rightarrow\ \underbrace{\forall y'.\ y' > 0 \ \rightarrow\ rem(x', y') < y'}_{F[x']} \qquad \text{(IH)}$$

Prove $\quad F[x] : \forall y.\ y > 0 \ \rightarrow\ rem(x, y) < y$.

Let y be an arbitrary positive integer

Case $x < y$:

$$\begin{array}{lcll} rem(x, y) & = & x & \text{by (R0)} \\ & < & y & \text{case} \end{array}$$

Case $\neg(x < y)$:

Then there is natural number n, $n < x$ s.t. $x = n + y$

$$\begin{array}{lcll} rem(x, y) & = & rem(n + y, y) & x = n + y \\ & = & rem(n, y) & \text{(R1)} \\ & < & y & \text{IH } (x' \mapsto n, y' \mapsto y) \\ & & & \text{since } n < x \text{ and } y > 0 \end{array}$$

Well-founded Induction

A binary predicate $\prec$ over a set S is a well-founded relation iff there does not exist an infinite decreasing sequence

$$s_1 \succ s_2 \succ s_3 \succ \cdots$$

Note: where $s \prec t$ iff $t \succ s$

Examples:

- $<$ is well-founded over the natural numbers.
  Any sequence of natural numbers decreasing according to $<$ is finite:
  $$1023 > 39 > 30 > 29 > 8 > 3 > 0.$$
- $<$ is not well-founded over the rationals.
  $$1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \cdots$$
  is an infinite decreasing sequence.
- The strict sublist relation $\prec_c$ is well-founded on the set of all lists.

Well-founded Induction Principle

For theory T and well-founded relation $\prec$,
the axiom schema (well-founded induction)

$$(\forall n.\ (\forall n'.\ n' \prec n \ \rightarrow\ F[n']) \ \rightarrow\ F[n]) \ \rightarrow\ \forall x.\ F[x]$$

for Σ-formulae $F[x]$ with one free variable x.

To prove $\forall x.\ F[x]$, i.e.,
$F[x]$ is T-valid for every x,
it suffices to show

- inductive step: For arbitrary n,
  assume inductive hypothesis, i.e.,
  $F[n']$ is T-valid for every n', such that $n' \prec n$
  then prove
  $F[n]$ is T-valid.

Complete induction in T_{PA} is a specific instance of well-founded induction, where the well-founded relation $\prec$ is $<$.

Lexicographic Relation

Given pairs of sets and well-founded relations

$$(S_1, \prec_1), \ldots, (S_m, \prec_m)$$

Construct

$$S = S_1 \times \ldots, S_m$$

Define lexicographic relation $\prec$ over S as

$$\underbrace{(s_1, \ldots, s_m)}_{s} \prec \underbrace{(t_1, \ldots, t_m)}_{t} \ \Leftrightarrow\ \bigvee_{i=1}^{m} \left(s_i \prec_i t_i \ \wedge\ \bigwedge_{j=1}^{i-1} s_j = t_j \right)$$

for $s_i, t_i \in S_i$.

- If $(S_1, \prec_1), \ldots, (S_m, \prec_m)$ are well-founded relations, so is $(S, \prec)$.

Lexicographic well-founded induction principle

For theory T and well-founded lexicographic relation $\prec$,

$$\left[\begin{array}{l}\forall n_1,\dots,n_m.\\ \left[\begin{array}{l}(\forall n'_1,\dots,n'_m.\ (n'_1,\dots,n'_m) \prec (n_1,\dots,n_m) \ \rightarrow\ F[n'_1,\dots,n'_m])\\ \quad\rightarrow\ F[n_1,\dots,n_m]\end{array}\right]\end{array}\right]$$
$$\rightarrow\ \forall x_1,\dots,x_m.\ F[x_1,\dots,x_m]$$

for Σ-formula $F[x_1,\dots,x_m]$ with free variables $x_1,\dots,x_m$, is T-valid.

Same as regular well-founded induction, just

$n \ \Rightarrow\ $ tuple $(n_1,\dots,n_m)$.

Example: Puzzle

Bag of red, yellow, and blue chips
If one chip remains in the bag – remove it
Otherwise, remove two chips at random:

1. If one of the two is red –
   don't put any chips in the bag
2. If both are yellow –
   put one yellow and five blue chips
3. If one of the two is blue and the other not red –
   put ten red chips

Does this process terminate?

Proof: Consider

- Set $S : \mathbb{N}^3$ of triples of natural numbers and
- Well-founded lexicographic relation $<_3$ for such triples, e.g.

$$(11, 13, 3) \not<_3 (11, 9, 104) \qquad (11, 9, 104) <_3 (11, 13, 3)$$

Show

$$(y', b', r') <_3 (y, b, r)$$

for each possible case. Since $<_3$ well-formed relation
$\Rightarrow$ only finite decreasing sequences $\Rightarrow$ process must terminate

1. If one of the two removed chips is red –
   do not put any chips in the bag
   $$\left.\begin{array}{c}(y-1, b, r-1)\\ (y, b-1, r-1)\\ (y, b, r-2)\end{array}\right\} <_3 (y, b, r)$$
2. If both are yellow –
   put one yellow and five blue
   $$(y-1, b+5, r) <_3 (y, b, r)$$
3. If one is blue and the other not red –
   put ten red
   $$\left.\begin{array}{c}(y-1, b-1, r+10)\\ (y, b-2, r+10)\end{array}\right\} <_3 (y, b, r)$$

Example: Ackermann function

Theory $T_{\mathbb{N}}^{ack}$ is the theory of Presburger arithmetic $T_{\mathbb{N}}$ (for natural numbers) augmented with

Ackermann axioms:

- $\forall y.\ ack(0, y) = y + 1$ (L0)
- $\forall x.\ ack(x+1, 0) = ack(x, 1)$ (R0)
- $\forall x, y.\ ack(x+1, y+1) = ack(x, ack(x+1, y))$ (S)

Ackermann function grows quickly:

$ack(0,0) = 1$
$ack(1,1) = 3$
$ack(2,2) = 7$
$ack(3,3) = 61$

$$ack(4,4) = 2^{2^{2^{16}}} - 3$$

Let $<_2$ be the lexicographic extension of $<$ to pairs of natural numbers.

(L0) $\forall y.\ ack(0, y) = y + 1$

does not involve recursive call

(R0) $\forall x.\ ack(x + 1, 0) = ack(x, 1)$

$$(x + 1, 0) >_2 (x, 1)$$

(S) $\forall x, y.\ ack(x + 1, y + 1) = ack(x, ack(x + 1, y))$

$$(x + 1, y + 1) >_2 (x + 1, y)$$
$$(x + 1, y + 1) >_2 (x, ack(x + 1, y))$$

No infinite recursive calls $\Rightarrow$ the recursive computation of $ack(x, y)$ terminates for all pairs of natural numbers.

Proof of property

Use well-founded induction over $<_2$ to prove

$$\forall x, y.\ ack(x, y) > y$$

is $T_{\mathbb{N}}^{ack}$ valid.

Consider arbitrary natural numbers x, y.

Assume the inductive hypothesis

$$\forall x', y'.\ \overline{(x', y') <_2 (x, y)} \ \rightarrow\ \underbrace{ack(x', y') > y'}_{F[x', y']} \qquad \text{(IH)}$$

Show

$$F[x, y] : ack(x, y) > y.$$

Case $x = 0$:

$$ack(0, y) = y + 1 > y \qquad \text{by (L0)}$$

Case $x > 0 \ \wedge\ y = 0$:

$$ack(x, 0) = ack(x - 1, 1) \qquad \text{by (R0)}$$

Since

$$(\underbrace{x - 1}_{x'}, \underbrace{1}_{y'}) <_2 (x, y)$$

Then

$$ack(x - 1, 1) > 1 \qquad \text{by (IH) } (x' \mapsto x - 1, y' \mapsto 1)$$

Thus

$$ack(x, 0) = ack(x - 1, 1) > 1 > 0$$

Case $x > 0 \ \wedge\ y > 0$:

$$ack(x, y) = ack(x - 1, ack(x, y - 1)) \qquad \text{by (S)} \qquad (1)$$

Since

$$(\underbrace{x - 1}_{x'}, \underbrace{ack(x, y - 1)}_{y'}) <_2 (x, y)$$

Then

$$ack(x - 1, ack(x, y - 1)) > ack(x, y - 1) \qquad (2)$$

by (IH) $(x' \mapsto x - 1, y' \mapsto ack(x, y - 1))$.

Furthermore, since

$$(\underbrace{x}_{x'}, \underbrace{y - 1}_{y'}) <_2 (x, y)$$

then

$$ack(x, y - 1) > y - 1 \qquad (3)$$

By (1)–(3), we have

$$ack(x, y) \overset{(1)}{=} ack(x - 1, ack(x, y - 1)) \overset{(2)}{>} ack(x, y - 1) \overset{(3)}{>} y - 1$$

Hence

$$ack(x, y) > (y - 1) + 1 = y$$

Structural Induction

How do we prove properties about logical formulae themselves?

Structural induction principle

To prove a desired property of FOL formulae,

inductive step: Assume the inductive hypothesis, that for arbitrary FOL formula F, the desired property holds for every strict subformula G of F.
Then prove that F has the property.

Since atoms do not have strict subformulae, they are treated as base cases.

Example: Prove that

Every propositional formula F is equivalent to a propositional formula F' constructed with only $\top$, $\vee$, $\neg$ (and propositional variables)

Base cases:

$F : \top \;\Rightarrow\; F' : \top$
$F : \bot \;\Rightarrow\; F' : \neg\top$
$F : P \;\Rightarrow\; F' : P$ for propositional variable P

Inductive step:

Assume as the inductive hypothesis that G, G_1, G_2 are equivalent to G', G_1', G_2' constructed only from $\top$, $\vee$, $\neg$ (and propositional variables).

$F : \neg G \quad\Rightarrow\quad F' : \neg G'$
$F : G_1 \wedge G_2 \quad\Rightarrow\quad F' : \neg(\neg G_1' \vee \neg G_2')$
$F : G_1 \rightarrow G_2 \quad\Rightarrow\quad F' : \neg G_1' \vee G_2'$
$F : G_1 \leftrightarrow G_2 \quad\Rightarrow\quad F' : \cdots$

Each F' is equivalent to F and is constructed only by $\top$, $\vee$, $\neg$ by the inductive hypothesis.