

Part II: Algorithm Reasoning

7. Quantified Linear Arithmetic

Quantifier Elimination (QE) — algorithm for elimination of all quantifiers of formula F until quantifier-free formula G that is equivalent to F remains

Note: Could be enough F is equisatisfiable to F' , that is F is satisfiable iff F' is satisfiable

A theory T admits quantifier elimination if there is an algorithm that given Σ -formula returns a quantifier-free Σ -formula G that is T -equivalent



Example

For $\Sigma_{\mathbb{Q}}$ -formula

$$F : \exists x. 2x = y,$$

quantifier-free $T_{\mathbb{Q}}$ -equivalent $\Sigma_{\mathbb{Q}}$ -formula is

$$G : \top$$

For $\Sigma_{\mathbb{Z}}$ -formula

$$F : \exists x. 2x = y,$$

there is no quantifier-free $T_{\mathbb{Z}}$ -equivalent $\Sigma_{\mathbb{Z}}$ -formula.

Let $T_{\hat{\mathbb{Z}}}$ be $T_{\mathbb{Z}}$ with divisibility predicates.

For $\Sigma_{\hat{\mathbb{Z}}}$ -formula

$$F : \exists x. 2x = y,$$

a quantifier-free $T_{\hat{\mathbb{Z}}}$ -equivalent $\Sigma_{\hat{\mathbb{Z}}}$ -formula is

$$G : 2 \mid y.$$



In developing a QE algorithm for theory T , we need only consider formulae of the form

$$\exists x. F$$

for quantifier-free F

Example: For Σ -formula

$$G_1: \exists x. \forall y. \underbrace{\exists z. F_1[x, y, z]}_{F_2[x, y]}$$

$$G_2: \exists x. \forall y. F_2[x, y]$$

$$G_3: \exists x. \underbrace{\neg \exists y. \neg F_2[x, y]}_{F_3[x]}$$

$$G_4: \underbrace{\exists x. \neg F_3[x]}_{F_4}$$

$$G_5: F_4$$

G_5 is quantifier-free and T -equivalent to G_1



Quantifier Elimination for $T_{\mathbb{Z}}$

$\Sigma_{\mathbb{Z}} : \{\dots, -2, -1, 0, 1, 2, \dots, -3, -2, 2, 3, \dots, +, -, =, <\}$

Lemma:

Given quantifier-free $\Sigma_{\mathbb{Z}}$ -formula F s.t. $\text{free}(F) = \{y\}$.
 F represents the set of integers

$$S : \{n \in \mathbb{Z} : F\{y \mapsto n\} \text{ is } T_{\mathbb{Z}}\text{-valid}\} .$$

Either $S \cap \mathbb{Z}^+$ or $\mathbb{Z}^+ \setminus S$ is finite.
 where \mathbb{Z}^+ is the set of positive integers

Example: $\Sigma_{\mathbb{Z}}$ -formula $F : \exists x. 2x = y$
 S : even integers

$S \cap \mathbb{Z}^+$: positive even integers — infinite
 $\mathbb{Z}^+ \setminus S$: positive odd integers — infinite

Therefore, by the lemma, there is no quantifier-free $T_{\mathbb{Z}}$ -formula that is $T_{\mathbb{Z}}$ -equivalent to F .

Thus, $T_{\mathbb{Z}}$ does not admit QE.

Augmented theory $\widehat{T}_{\mathbb{Z}}$

$\widehat{\Sigma}_{\mathbb{Z}}$: $\Sigma_{\mathbb{Z}}$ with countable number of unary divisibility predicates

$$k \mid \cdot \quad \text{for } k \in \mathbb{Z}^+$$

Intended interpretations:

$k \mid x$ holds iff k divides x without any remainder

Example:

$$x > 1 \wedge y > 1 \wedge 2 \mid x + y$$

is satisfiable (choose $x = 2, y = 2$).

$$\neg(2 \mid x) \wedge 4 \mid x$$

is not satisfiable.

Axioms of $\widehat{T}_{\mathbb{Z}}$: axioms of $T_{\mathbb{Z}}$ with additional countable set of axioms

$$\forall x. k \mid x \leftrightarrow \exists y. x = ky \quad \text{for } k \in \mathbb{Z}^+$$

$\widehat{T}_{\mathbb{Z}}$ admits QE (Cooper's method)

Algorithm: Given $\widehat{\Sigma}_{\mathbb{Z}}$ -formula $\exists x. F[x]$, where F is quantifier-free
 Construct quantifier-free $\widehat{\Sigma}_{\mathbb{Z}}$ -formula that is equivalent to $\exists x. F[x]$.

Step 1

Put $F[x]$ in NNF $F_1[x]$, that is,
 $\exists x. F_1[x]$ has negations only in literals (only \wedge, \vee)
 and $\widehat{T}_{\mathbb{Z}}$ -equivalent to $\exists x. F[x]$

Step 2

Replace (left to right)

$$\begin{aligned} s = t &\Leftrightarrow s < t + 1 \wedge t < s + 1 \\ \neg(s = t) &\Leftrightarrow s < t \vee t < s \\ \neg(s < t) &\Leftrightarrow t < s + 1 \end{aligned}$$

The output $\exists x. F_2[x]$ contains only literals of form

$$s < t, \quad k \mid t, \quad \text{or} \quad \neg(k \mid t),$$

where s, t are $\widehat{T}_{\mathbb{Z}}$ -terms and $k \in \mathbb{Z}^+$.

Example:

$$\begin{aligned} \neg(x < y) \wedge \neg(x = y + 3) \\ \Downarrow \\ y < x + 1 \wedge (x < y + 3 \vee y + 3 < x) \end{aligned}$$

Step 3

Collect terms containing x so that literals have the form

$$hx < t, \quad t < hx, \quad k \mid hx + t, \quad \text{or} \quad \neg(k \mid hx + t),$$

where t is a term and $h, k \in \mathbb{Z}^+$. The output is the formula $\exists x. F_3[x]$, which is $\widehat{T}_{\mathbb{Z}}$ -equivalent to $\exists x. F[x]$.

Example:

$$\begin{aligned} x + x + y < z + 3z + 2y - 4x \\ \Downarrow \\ 6x < 4z + y \end{aligned}$$

Step 4

Let

$$\delta' = \text{lcm}\{h : h \text{ is a coefficient of } x \text{ in } F_3[x]\},$$

where lcm is the least common multiple. Multiply atoms in $F_3[x]$ by constants so that δ' is the coefficient of x everywhere:

$$\begin{aligned} hx < t &\Leftrightarrow \delta'x < h't && \text{where } h'h = \delta' \\ t < hx &\Leftrightarrow h't < \delta'x && \text{where } h'h = \delta' \\ k \mid hx + t &\Leftrightarrow h'k \mid \delta'x + h't && \text{where } h'h = \delta' \\ \neg(k \mid hx + t) &\Leftrightarrow \neg(h'k \mid \delta'x + h't) && \text{where } h'h = \delta' \end{aligned}$$

The result $\exists x. F_3'[x]$, in which all occurrences of x in $F_3'[x]$ are in terms $\delta'x$.

Replace $\delta'x$ terms in F_3' with a fresh variable x' to form

$$F_3'' : F_3\{\delta'x \mapsto x'\}$$

Finally, construct

$$\exists x'. \underbrace{F_3''[x'] \wedge \delta' \mid x'}_{F_4[x']}$$

$\exists x'. F_4[x']$ is equivalent to $\exists x. F[x]$ and each literal of $F_4[x']$ has one of the forms:

- (A) $x' < a$
- (B) $b < x'$
- (C) $h \mid x' + c$
- (D) $\neg(k \mid x' + d)$

where a, b, c, d are terms that do not contain x , and $h, k \in \mathbb{Z}^+$.

Example: $\widehat{T}_{\mathbb{Z}}$ -formula

$$\exists x. \underbrace{3x + 1 > y \wedge 2x - 6 < z \wedge 4 \mid 5x + 1}_{F[x]}$$

after step 3

$$\exists x. \underbrace{2x < z + 6 \wedge y - 1 < 3x \wedge 4 \mid 5x + 1}_{F_3[x]}$$

Collecting coefficients of x (step 4),

$$\delta' = \text{lcm}(2, 3, 5) = 30$$

Multiply when necessary

$$\exists x. 30x < 15z + 90 \wedge 10y - 10 < 30x \wedge 24 \mid 30x + 6$$

Replacing $30x$ with fresh x'

$$\exists x'. \underbrace{x' < 15z + 90 \wedge 10y - 10 < x' \wedge 24 \mid x' + 6 \wedge 30 \mid x'}_{F_4[x']}$$

$\exists x'. F_4[x']$ is equivalent to $\exists x. F[x]$

Step 5 (trickiest part):

Construct

left infinite projection $F_{-\infty}[x']$

of $F_4[x']$ by

- (A) replacing literals $x' < a$ by \top
- (B) replacing literals $b < x'$ by \perp

idea: very small numbers satisfy (A) literals but not (B) literals

Let

$$\delta = \text{lcm} \left\{ \begin{array}{l} h \text{ of (C) literals } h \mid x' + c \\ k \text{ of (D) literals } \neg(k \mid x' + d) \end{array} \right\}$$

and B be the set of b terms appearing in (B) literals. Construct

$$F_5 : \bigvee_{j=1}^{\delta} F_{-\infty}[j] \vee \bigvee_{j=1}^{\delta} \bigvee_{b \in B} F_4[b + j].$$

F_5 is quantifier-free and $\widehat{T}_{\mathbb{Z}}$ -equivalent to F .

Intuition

Property (Periodicity)

if $k \mid \delta$

then $k \mid n$ iff $k \mid n + \lambda\delta$ for all $\lambda \in \mathbb{Z}$

That is, $k \mid \cdot$ cannot distinguish between $k \mid n$ and $k \mid n + \lambda\delta$.

By the choice of δ (lcm of the h 's and k 's) — no \mid literal in F_5 can distinguish between n and $n + \delta$.

$$F_5 : \bigvee_{j=1}^{\delta} F_{-\infty}[j] \vee \bigvee_{j=1}^{\delta} \bigvee_{b \in B} F_4[b + j]$$

left disjunct $\bigvee_{j=1}^{\delta} F_{-\infty}[j]$:

Contains only \mid literals

Asserts: no least $n \in \mathbb{Z}$ s.t. $F[n]$.

For if there exists n satisfying $F_{-\infty}$,

then every $n - \lambda\delta$, for $\lambda \in \mathbb{Z}^+$, also satisfies $F_{-\infty}$

right disjunct $\bigvee_{j=1}^{\delta} \bigvee_{b \in B} F_4[b + j]$:

Asserts: There is least $n \in \mathbb{Z}$ s.t. $F[n]$.

For let b^* be the largest b in (B) .

If $n \in \mathbb{Z}$ is s.t. $F[n]$,

then

$$\exists j (1 \leq j \leq \delta). b^* + j \leq n \wedge F[b^* + j]$$

In other words,

if there is a solution,

then one must appear in δ interval to the right of b^*

Example (cont):

$$\begin{array}{c} \exists x. \underbrace{3x + 1 > y \wedge 2x - 6 < z \wedge 4 \mid 5x + 1}_{F[x]} \\ \downarrow \\ \exists x'. \underbrace{x' < 15z + 90 \wedge 10y - 10 < x' \wedge 24 \mid x' + 6 \wedge 30 \mid x'}_{F_4[x']} \end{array}$$

By step 5,

$$F_{-\infty}[x] : \top \wedge \perp \wedge 24 \mid x' + 6 \wedge 30 \mid x',$$

which simplifies to \perp . Compute

$$\delta = \text{lcm}\{24, 30\} = 120 \quad \text{and} \quad B = \{10y - 10\}.$$

Then replacing x' by $10y - 10 + j$ in $F_4[x']$ produces

$$F_5 : \bigvee_{j=1}^{120} \left[\begin{array}{l} 10y - 10 + j < 15z + 90 \wedge 10y - 10 < 10y - 10 + j \\ \wedge 24 \mid 10y - 10 + j + 6 \wedge 30 \mid 10y - 10 + j \end{array} \right]$$

which simplifies to

$$F_5 : \bigvee_{j=1}^{120} \left[\begin{array}{l} 10y + j < 15z + 100 \wedge 0 < j \\ \wedge 24 \mid 10y + j - 4 \wedge 30 \mid 10y - 10 + j \end{array} \right].$$

F_5 is quantifier-free and $\widehat{T}_{\mathbb{Z}}$ -equivalent to F .

Example:

$$\exists x. \underbrace{(3x + 1 < 10 \vee 7x - 6 > 7) \wedge 2 \mid x}_{F[x]}$$

Isolate x terms

$$\exists x. (3x < 9 \vee 13 < 7x) \wedge 2 \mid x,$$

so

$$\delta' = \text{lcm}\{3, 7\} = 21.$$

After multiplying coefficients by proper constants,

$$\exists x. (21x < 63 \vee 39 < 21x) \wedge 42 \mid 21x,$$

we replace $21x$ by x' :

$$\exists x'. \underbrace{(x' < 63 \vee 39 < x') \wedge 42 \mid x' \wedge 21 \mid x'}_{F_4[x']}.$$

Then

$$F_{-\infty}[x'] : (\top \vee \perp) \wedge 42 \mid x' \wedge 21 \mid x' ,$$

or, simplifying,

$$F_{-\infty}[x'] : 42 \mid x' \wedge 21 \mid x' .$$

Finally,

$$\delta = \text{lcm}\{21, 42\} = 42 \quad \text{and} \quad B = \{39\} ,$$

so

$$F_5 : \bigvee_{j=1}^{42} (42 \mid j \wedge 21 \mid j) \vee \bigvee_{j=1}^{42} ((39 + j < 63 \vee 39 < 39 + j) \wedge 42 \mid 39 + j \wedge 21 \mid 39 + j) .$$

Since $42 \mid 42$ and $21 \mid 42$, the left main disjunct simplifies to \top , so that F is $\widehat{T}_{\mathbb{Z}}$ -equivalent to \top . Thus, F is $\widehat{T}_{\mathbb{Z}}$ -valid.

However,

$$\delta = \text{lcm}\{2\} = 2 \quad \text{and} \quad B = \{y - 1\} ,$$

so

$$F_5 : \bigvee_{j=1}^2 (y - 1 < y - 1 + j \wedge y - 1 + j < y + 1 \wedge 2 \mid y - 1 + j)$$

Simplifying,

$$F_5 : \bigvee_{j=1}^2 (0 < j \wedge j < 2 \wedge 2 \mid y - 1 + j)$$

and then

$$F_5 : 2 \mid y ,$$

which is quantifier-free and $\widehat{T}_{\mathbb{Z}}$ -equivalent to F .

Example:

$$\exists x. \underbrace{2x = y}_{F[x]}$$

Rewriting

$$\exists x. \underbrace{y - 1 < 2x \wedge 2x < y + 1}_{F_3[x]}$$

Then

$$\delta' = \text{lcm}\{2, 2\} = 2 ,$$

so by Step 4

$$\exists x'. \underbrace{y - 1 < x' \wedge x' < y + 1 \wedge 2 \mid x'}_{F_4[x']}$$

$F_{-\infty}$ produces \perp .

Two Improvements:

A. Symmetric Elimination

In step 5, if there are fewer

(A) literals $x' < a$

than

(B) literals $b < x'$.

Construct the right infinite projection $F_{+\infty}[x']$ from $F_4[x']$ by replacing

each (A) literal $x' < a$ by \perp

and

each (B) literal $b < x'$ by \top .

Then right elimination.

$$F_5 : \bigvee_{j=1}^{\delta} F_{+\infty}[-j] \vee \bigvee_{j=1}^{\delta} \bigvee_{a \in A} F_4[a - j] .$$

B. Eliminating Blocks of Quantifiers

$$\exists x_1. \dots \exists x_n. F[x_1, \dots, x_n]$$

where F quantifier-free.

Eliminating x_n (left elimination) produces

$$G_1 : \exists x_1. \dots \exists x_{n-1}. \bigvee_{j=1}^{\delta} F_{-\infty}[x_1, \dots, x_{n-1}, j] \vee \bigvee_{j=1}^{\delta} \bigvee_{b \in B} F_4[x_1, \dots, x_{n-1}, b + j]$$

which is equivalent to

$$G_2 : \bigvee_{j=1}^{\delta} \exists x_1. \dots \exists x_{n-1}. F_{-\infty}[x_1, \dots, x_{n-1}, j] \vee \bigvee_{j=1}^{\delta} \bigvee_{b \in B} \exists x_1. \dots \exists x_{n-1}. F_4[x_1, \dots, x_{n-1}, b + j]$$

Treat j as a free variable and examine only $1 + |B|$ formulae

- ▶ $\exists x_1. \dots \exists x_{n-1}. F_{-\infty}[x_1, \dots, x_{n-1}, j]$
- ▶ $\exists x_1. \dots \exists x_{n-1}. F_4[x_1, \dots, x_{n-1}, b + j]$ for each $b \in B$

Apply QE (treating j as free variable)

$$H : \exists y. j > 0 \wedge 39 + j < 65y \wedge y < -27 - j \wedge 13 \mid -26 - j$$

Simplify

$$H' : \bigvee_{k=1}^{65} (k < -1794 - 66j \wedge 13 \mid -26 - j \wedge 65 \mid 39 + j + k)$$

Replace H with H' in G

$$\bigvee_{j=1}^{13} \bigvee_{k=1}^{65} (k < -1794 - 66j \wedge 13 \mid -26 - j \wedge 65 \mid 39 + j + k)$$

This formula is $\widehat{T}_{\mathbb{Z}}$ -equivalent to F .

Example:

$$F : \exists y. \exists x. x < -2 \wedge 1 - 5y < x \wedge 1 + y < 13x$$

Since $\delta' = \text{lcm}\{1, 13\} = 13$

$$\exists y. \exists x. 13x < -26 \wedge 13 - 65y < 13x \wedge 1 + y < 13x$$

Then

$$\exists y. \exists x'. x' < -26 \wedge 13 - 65y < x' \wedge 1 + y < x' \wedge 13 \mid x'$$

There is one (A) literal $x' < \dots$ and two (B) literals $\dots < x'$, we use right elimination.

$$F_{+\infty} = \perp \quad \delta = \{13\} = 13 \quad A = \{-26\}$$

$$\exists y. \bigvee_{j=1}^{13} \left[\begin{array}{l} -26 - j < -26 \wedge 13 - 65y < -26 - j \\ \wedge 1 + y < -26 - j \wedge 13 \mid -26 - j \end{array} \right]$$

Commute

$$G : \bigvee_{j=1}^{13} \exists y. j > 0 \wedge 39 + j < 65y \wedge y < -27 - j \wedge 13 \mid -26 - j$$

Quantifier Elimination over Rationals

$$\Sigma_{\mathbb{Q}} : \{0, 1, +, -, =, \geq\}$$

we use $>$ instead of \geq , as

$$x \geq y \Leftrightarrow x > y \vee x = y \quad x > y \Leftrightarrow x \geq y \wedge \neg(x = y).$$

Ferrante and Rackoff's Method

Given a $\Sigma_{\mathbb{Q}}$ -formula $\exists x. F[x]$, where $F[x]$ is quantifier-free

Generate quantifier-free formula F_4 (four steps) s.t.

$$F_4 \text{ is } \Sigma_{\mathbb{Q}}\text{-equivalent to } \exists x. F[x].$$

Step 1: Put $F[x]$ in NNF. The result is $\exists x. F_1[x]$.

Step 2: Replace literals (left to right)

$$\begin{aligned} \neg(s < t) &\Leftrightarrow t < s \vee t = s \\ \neg(s = t) &\Leftrightarrow t < s \vee t > s \end{aligned}$$

The result $\exists x. F_2[x]$ does not contain negations.

Step 3: Solve for x in each atom of $F_2[x]$, e.g.,

$$t < cx \quad \Rightarrow \quad \frac{t}{c} < x$$

where $c \in \mathbb{Z} - \{0\}$.

All atoms in the result $\exists x. F_3[x]$ have form

(A) $x < a$

(B) $b < x$

(C) $x = c$

where a, b, c are terms that do not contain x .

Step 4: Construct from $F_3[x]$

▶ left infinite projection $F_{-\infty}$ by replacing

(A) atoms $x < a$ by \top

(B) atoms $b < x$ by \perp

(C) atoms $x = c$ by \perp

▶ right infinite projection $F_{+\infty}$ by replacing

(A) atoms $x < a$ by \perp

(B) atoms $b < x$ by \top

(C) atoms $x = c$ by \perp

Let S be the set of a, b, c terms from (A), (B), (C) atoms.

Construct the final

$$F_4 : F_{-\infty} \vee F_{+\infty} \vee \bigvee_{s,t \in S} F_3 \left[\frac{s+t}{2} \right],$$

which is $T_{\mathbb{Q}}$ -equivalent to $\exists x. F[x]$.

▶ $F_{-\infty}$ captures the case when small $n \in \mathbb{Q}$ satisfy $F_3[n]$

▶ $F_{+\infty}$ captures the case when large $n \in \mathbb{Q}$ satisfy $F_3[n]$

▶ last disjunct: for $s, t \in S$

if $s \equiv t$, check whether $s \in S$ satisfies $F_4[s]$

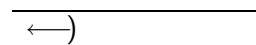
if $s \not\equiv t$, $\frac{s+t}{2}$ represents the whole interval (s, t) , so check

$$F_4 \left[\frac{s+t}{2} \right]$$

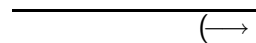
Intuition

Step 4 says that four cases are possible:

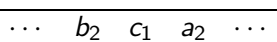
1. There is a left open interval s.t. all elements satisfy $F(x)$.



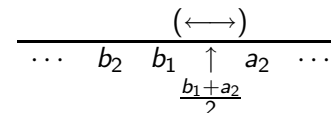
2. There is a right open interval s.t. all elements satisfy $F(x)$.



3. Some $a_i, b_i,$ or c_i satisfies $F(x)$.



4. There is an open interval between two $a_i, b_i,$ or c_i terms s.t. every element satisfies $F(x)$.



Example: $\Sigma_{\mathbb{Q}}$ -formula

$$\exists x. \underbrace{3x + 1 < 10 \wedge 7x - 6 > 7}_{F[x]}$$

Solving for x

$$\exists x. \underbrace{x < 3 \wedge x > \frac{13}{7}}_{F_3[x]}$$

$$\begin{array}{l} \text{Step 4: } x < 3 \text{ in (A)} \Rightarrow F_{-\infty} = \perp \\ x > \frac{13}{7} \text{ in (B)} \Rightarrow F_{+\infty} = \perp \end{array}$$

$$F_4 : \bigvee_{s,t \in S} \underbrace{\left(\frac{s+t}{2} < 3 \wedge \frac{s+t}{2} > \frac{13}{7} \right)}_{F_3[\frac{s+t}{2}]}$$

$$S = \{3, \frac{13}{7}\} \Rightarrow$$

$$F_3 \left[\frac{3+3}{2} \right] = \perp \quad F_3 \left[\frac{\frac{13}{7} + \frac{13}{7}}{2} \right] = \perp$$

$$F_3 \left[\frac{\frac{13}{7} + 3}{2} \right] : \frac{\frac{13}{7} + 3}{2} < 3 \wedge \frac{\frac{13}{7} + 3}{2} > \frac{13}{7}$$

simplifies to \top .

Thus, $F_4 : \top$ is $T_{\mathbb{Q}}$ -equivalent to $\exists x. F[x]$,
so $\exists x. F[x]$ is $T_{\mathbb{Q}}$ -valid.