

8. Quantifier-Free Linear Arithmetic

Decision Procedures for Quantifier-free Fragments

For theory T with signature Σ and axioms Σ -formulae of form $\forall x_1, \dots, x_n. F[x_1, \dots, x_n]$

Decide if

$F[x_1, \dots, x_n]$ or $\exists x_1, \dots, x_n. F[x_1, \dots, x_n]$ is T -satisfiable

[Decide if $F[x_1, \dots, x_n]$ or $\forall x_1, \dots, x_n. F[x_1, \dots, x_n]$ is T -valid]

where F is quantifier-free and $\text{free}(F) = \{x_1, \dots, x_n\}$

Note: no quantifier alternations

We consider only conjunctive quantifier-free Σ -formulae, i.e., conjunctions of Σ -literals (Σ -atoms or negations of Σ -atoms). For given arbitrary quantifier-free Σ -formula F , convert it into DNF Σ -formula

$$F_1 \vee \dots \vee F_k$$

where each F_i conjunctive.

F is T -satisfiable iff at least one F_i is T -satisfiable.



Preliminary Concepts

Vector

variable n -vector n -vector $\bar{a} \in \mathbb{Q}^n$ transpose

$$\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \bar{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \bar{a}^T = [a_1 \ \cdots \ a_n]$$

Matrix

$m \times n$ -matrix $A \in \mathbb{Q}^{m \times n}$ transpose

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad A^T = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix}$$

column \rightarrow a_{1j}
 \rightarrow row $a_{i1} \cdots a_{ij} \cdots a_{in}$
 a_{mj}



Multiplication

vector-vector

$$\bar{a}^T \bar{b} = [a_1 \ \cdots \ a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^n a_i b_i$$

matrix-vector

$$A \bar{x} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_{1i} x_i \\ \vdots \\ \sum_{i=1}^n a_{mi} x_i \end{bmatrix}$$

matrix-matrix

$$\begin{bmatrix} a_{ik} \\ \vdots \\ a_{mk} \end{bmatrix} \begin{bmatrix} b_{kj} \\ \vdots \\ b_{nj} \end{bmatrix} = \begin{bmatrix} p_{ij} \\ \vdots \\ p_{mj} \end{bmatrix}$$

$A \quad B \quad P$

where $p_{ij} = \bar{a}_i \bar{b}_j = [a_{i1} \ \cdots \ a_{in}] \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix} = \sum_{k=1}^n a_{ik} b_{kj}$



Special Vectors and Matrices

$\vec{0}$ - vector (column) of 0s

$\vec{1}$ - vector of 1s

$$\text{Thus } \vec{1}^T \vec{x} = \sum_{i=1}^n x_i$$

$$I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \text{ identity matrix } (n \times n)$$

Thus $IA = AI = A$

$$\text{unit vector } e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$$

Vector Space - set S of vectors closed under addition and scaling of vectors. That is,

$$\text{if } \vec{v}_1, \dots, \vec{v}_k \in S \text{ then } \lambda_1 \vec{v}_1 + \dots + \lambda_k \vec{v}_k \in S$$

$$\text{for } \lambda_1, \dots, \lambda_n \in \mathbb{R}.$$

Linear Equation

$$F : A\vec{x} = \vec{b}$$

$m \times n$ -matrix variable n -vector m -vector

represents the $\Sigma_{\mathbb{Q}}$ -formula

$$F : (a_{11}x_1 + \dots + a_{1n}x_n = b_1) \wedge \dots \wedge (a_{m1}x_1 + \dots + a_{mn}x_n = b_m)$$

Gaussian Elimination

Find \vec{x} s.t. $A\vec{x} = \vec{b}$ by elementary row operations

- ▶ Swap two rows.
- ▶ Multiply a row by a nonzero scalar.
- ▶ Add one row to another.

Example:

Solve

$$\begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix}$$

Construct the augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 6 \\ 1 & 0 & 1 & 1 \\ 2 & 2 & 1 & 2 \end{array} \right]$$

Apply the row operations as follows:

1. Add $-2\vec{a}_1 + 4\vec{a}_2$ to \vec{a}_3

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 6 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -6 \end{array} \right]$$

2. Add $-\vec{a}_1 + 2\vec{a}_2$ to \vec{a}_2

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 6 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 1 & -6 \end{array} \right]$$

This augmented matrix is in triangular form.

Solving

$$x_3 = -6$$

$$-x_2 - x_3 = -3 \Rightarrow x_2 = -3$$

$$3x_1 + x_2 + 2x_3 = 6 \Rightarrow x_1 = 7$$

The solution is $\vec{x} = [7 \ -3 \ -6]^T$

Inverse Matrix

A^{-1} is the inverse matrix of square matrix A if

$$AA^{-1} = A^{-1}A = I$$

Square matrix A is nonsingular (invertible) if its inverse A^{-1} exists.

How to compute A^{-1} of A ?

$$[A \mid I] \xrightarrow{\substack{\text{elementary} \\ \text{row operations}}} [I \mid A^{-1}]$$

How to compute k th column of A^{-1} ?

Solve $A\bar{y} = e_k$, i.e.

$$\left[A \begin{array}{c} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{array} \right] \xrightarrow{\substack{\text{elementary} \\ \text{row operations}}} \bar{y} = \dots \quad (\text{kth column of } A^{-1})$$

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Linear Inequality

$$G : A\bar{x} \leq b$$

represents the $\Sigma_{\mathbb{Q}}$ -formula

$$G : (a_{11}x_1 + \dots + a_{1n}x_n \leq b_1) \wedge \dots \wedge (a_{m1}x_1 + \dots + a_{mn}x_n \leq b_m)$$

The inequality describes a polyhedron in \mathbb{R}^n .

For $m \times n$ -matrix A , m -vector b , variable n -vector \bar{x} where $m \geq n$:

An n -vector \bar{v} is a vertex of $A\bar{x} \leq b$ if there is nonsingular $n \times n$ -submatrix A_0 and corresponding n -subvector b_0 s.t.

$$A_0\bar{v} = b_0$$

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Optimization Problem

$$\begin{array}{ll} \max & \bar{c}^T \bar{x} \quad \dots \text{objective function} \\ \text{subject to} & \\ & A\bar{x} \leq \bar{b} \quad \dots \text{constraints} \end{array}$$

Solution: vertex \bar{v}^* satisfying $A\bar{x} \leq \bar{b}$ and maximize $\bar{c}^T \bar{x}$. That is,

$$A\bar{v}^* \leq \bar{b} \text{ and}$$

$$\bar{c}^T \bar{v}^* \text{ is maximal: } \bar{c}^T \bar{v}^* \geq \bar{c}^T \bar{u} \text{ for all } \bar{u} \text{ satisfying } A\bar{u} \leq \bar{b}$$

- ▶ If $A\bar{x} \leq \bar{b}$ is unsatisfiable \Rightarrow maximum is $-\infty$
- ▶ It's possible that the maximum is unbounded \Rightarrow maximum is ∞

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Example: Consider optimization problem:

$$\begin{array}{ll} \max & \underbrace{\begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}}_{\bar{c}^T} \underbrace{\begin{bmatrix} x \\ y \\ z_1 \\ z_2 \end{bmatrix}}_{\bar{x}} \\ \text{subject to} & \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \\ z_1 \\ z_2 \end{bmatrix}}_{\bar{x}} \leq \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 2 \\ 2 \end{bmatrix}}_{\bar{b}} \end{array}$$

A is a 7×4 -matrix, \bar{b} is a 7-vector, and

\bar{x} is a variable 4-vector representing the four variables $\{x, y, z_1, z_2\}$.

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Example (cont):

The objective function is

$$(x - z_1) + (y - z_2).$$

The constraints are equivalent to the $\Sigma_{\mathbb{Q}}$ -formula

$$\begin{aligned} x \geq 0 \wedge y \geq 0 \wedge z_1 \geq 0 \wedge z_2 \geq 0 \\ \wedge x + y \leq 3 \wedge x - z_1 \leq 2 \wedge y - z_2 \leq 2 \end{aligned}$$

$\bar{v} = [2 \ 1 \ 0 \ 0]^T$ is a vertex of the constraints. For the nonsingular submatrix A_0 (rows 3, 4, 5, 6 of A), we have

$$\underbrace{\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}}_{A_0} \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\bar{v}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 3 \\ 2 \end{bmatrix}}_{b_0}$$

Duality Theorem

For $A \in \mathbb{Z}^{m \times n}$, $\bar{b} \in \mathbb{Z}^m$, $\bar{c} \in \mathbb{Z}^n$,

$$\max\{\bar{c}^T \bar{x} \mid A\bar{x} \leq \bar{b}\} = \min\{\bar{y}^T \bar{b} \mid \bar{y} \geq \bar{0} \wedge \bar{y}^T A = \bar{c}^T\}$$

if the constraints are satisfiable.

That is,

maximizing the function $c^T \bar{x}$ over $A\bar{x} \leq \bar{b}$
(the primal form of the optimization problem)

is equivalent to

minimizing the function $\bar{y}^T \bar{b}$ over all the nonnegative \bar{y}
s.t. $\bar{y}^T A = \bar{c}^T$
(the dual form of the optimization problem)

Outline of Algorithm

Given $\Sigma_{\mathbb{Q}}$ -formula

$$F : a_{11}x_1 + \dots + a_{1n}x_n \leq b_1 \wedge \dots \wedge a_{m1}x_1 + \dots + a_{mn}x_n \leq b_m \quad (7.1)$$

or in matrix notation

$$F : A\bar{x} \leq \bar{b}$$

Note: • equations

$$a_{i1}x_1 + \dots + a_{in}x_n = b_i$$

are allowed — break into two inequalities

$$a_{i1}x_1 + \dots + a_{in}x_n \leq b_i \wedge -a_{i1}x_1 - \dots - a_{in}x_n \leq -b_i.$$

• Strict inequalities

$$a_{i1}x_1 + \dots + a_{in}x_n < b_i.$$

excluded from our discussion — but can be added.

Outline of Algorithm (cont)

To determine the satisfiability of F ,

Step 0: reformulate the satisfiability of F as an optimization problem

$$M_F : \max\{\bar{c}^T \bar{x}' \mid A'\bar{x}' \leq \bar{b}'\}$$

s.t. F is $T_{\mathbb{Q}}$ -satisfiable iff the optimal value of M_F is a particular value v_F (derived from the structure of F)

Step 1, Step 2, ... (until termination) execute the simplex method

Outline of Algorithm (cont)

The simplex method traverses the vertices of $A'\bar{x}' \leq \bar{b}'$ searching for the maximum of the objective function $\bar{c}^T \bar{x}'$:
if $\bar{v}_1, \bar{v}_2, \dots$ are the traversed vertices in Step 1, Step 2, \dots , then

$$\bar{c}^T \bar{v}_1 < \bar{c}^T \bar{v}_2 < \dots$$

The simplex method terminates at some vertex \bar{v}_{j^*} where $\bar{c}^T \bar{v}_{j^*}$ is the global optimum

Final step: Compare the discovered optimal value $\bar{c}^T \bar{v}_{j^*}$ to the desired value v_F .

- ▶ if equal, then F is $T_{\mathbb{Q}}$ -satisfiable
- ▶ otherwise, F is $T_{\mathbb{Q}}$ -unsatisfiable

Step 0: From Satisfiability to Optimization

Given $\Sigma_{\mathbb{Q}}$ -formula

$$F : A\bar{x} \leq \bar{b} \tag{7.1}$$

reformulate to new constraint system (new A, \bar{x}, \bar{b})

$$F' : \bar{x} \geq 0, A\bar{x} \leq \bar{b}$$

F' is $T_{\mathbb{Q}}$ -equisatisfiable to F

The trick: replace each variable x in F by $x_1 - x_2$

Separate the new constraints $A\bar{x} \leq \bar{b}$ into

$$D_1\bar{x} \leq \bar{e}_1 \quad \text{and} \quad D_2\bar{x} \geq \bar{e}_2 \quad \text{for} \quad \bar{e}_1 \geq \bar{0}, \bar{e}_2 > \bar{0}$$

Example: $\Sigma_{\mathbb{Q}}$ -formula

$$F : x + y \geq 1 \wedge x - y \geq -1.$$

To convert it to the form $\bar{x} \geq \bar{0} \wedge A\bar{x} \leq \bar{b}$, introduce nonnegative x_1, x_2 for x and y_1, y_2 for y :

$$F' : \begin{aligned} (x_1 - x_2) + (y_1 - y_2) &\geq 1 \wedge (x_1 - x_2) - (y_1 - y_2) \geq -1 \\ \wedge x_1, x_2, y_1, y_2 &\geq 0 \end{aligned}$$

F is $T_{\mathbb{Q}}$ -equisatisfiable to F' . In matrix form,

$$F' : \underbrace{\begin{bmatrix} -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} \leq \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\bar{b}}$$

Since $b_1 < 0$ and $b_2 > 0$, separating constraints yields

$$\underbrace{\begin{bmatrix} -1 & 1 & 1 & -1 \end{bmatrix}}_{D_1} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} \leq \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{\bar{e}_1} \quad \text{and} \quad \underbrace{\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}}_{D_2} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} \geq \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{\bar{e}_2}$$

Generate the optimization problem:

$$M_F : \max \bar{1}^T (D_2\bar{x} - \bar{z}) \tag{7.2}$$

subject to

$$\bar{x}, \bar{z} \geq \bar{0} \tag{1}$$

$$D_1\bar{x} \leq \bar{e}_1 \tag{2}$$

$$D_2\bar{x} - \bar{z} \leq \bar{e}_2 \tag{3}$$

length of variable vector $\bar{z} = \#$ of rows of D_2

- ▶ The point $\bar{x} = \bar{0}, \bar{z} = \bar{0}$ satisfies constraints (1) – (3). It's a vertex.
- ▶ The optimum v_F equals $\bar{1}^T \bar{e}_2$ iff F is $T_{\mathbb{Q}}$ -satisfiable.

M_F can be written in standard form as

$$M_F : \max \underbrace{\bar{c}^T}_{\bar{c}^T} \underbrace{\begin{bmatrix} D_2 & -I \end{bmatrix}}_{\bar{y}} \begin{bmatrix} \bar{x} \\ \bar{z} \end{bmatrix} \quad (7.3)$$

subject to

$$\underbrace{\begin{bmatrix} -I & & & & \\ & -I & & & \\ & & D_1 & & \\ & & & & D_2 & -I \end{bmatrix}}_A \underbrace{\begin{bmatrix} \bar{x} \\ \bar{z} \end{bmatrix}}_{\bar{y}} \leq \underbrace{\begin{bmatrix} \bar{0} \\ \bar{0} \\ \bar{e}_1 \\ \bar{e}_2 \end{bmatrix}}_{\bar{b}}$$

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Example (cont):

Rewriting the optimization problem

$$\max \underbrace{\begin{bmatrix} 1 & -1 & 1 & -1 & -1 \end{bmatrix}}_{\bar{c}^T} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z \end{bmatrix}$$

subject to

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & -1 & -1 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z \end{bmatrix} \leq \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{\bar{b}}$$

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Example (cont):

$$\underbrace{\begin{bmatrix} -1 & 1 & 1 & -1 \end{bmatrix}}_{D_1} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} \leq \underbrace{[1]}_{\bar{e}_1} \quad \text{and} \quad \underbrace{\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}}_{D_2} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} \geq \underbrace{[1]}_{\bar{e}_2}$$

D_2 has only one row, so $\bar{z} = [z]$. Pose the following optimization problem:

$$\begin{aligned} \max \quad & \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} - [z] \\ \text{subject to} \quad & x_1, x_2, y_1, y_2, z \geq 0 \\ & \begin{bmatrix} -1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} \leq [1] \\ & \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} - [z] \leq [1] \end{aligned}$$

F is $T_{\mathbb{Q}}$ -satisfiable iff the optimum is $\bar{1}^T \bar{e}_2 = 1$.

$[x_1 \ x_2 \ y_1 \ y_2 \ z] = [0 \ 0 \ 0 \ 0 \ 0]$ is a vertex.

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Step i ($i \geq 1$): Find a Better Vertex

Optimization problem of form

$$\begin{aligned} \max \quad & \bar{c}^T \bar{x} \\ \text{subject to} \quad & A\bar{x} \leq \bar{b} \end{aligned}$$

we are given satisfying vertex \bar{v}_i .

- ▶ The simplex method traverses vertices of the space defined by $A\bar{x} \leq \bar{b}$ to find the vertex \bar{v}^* that maximizes $\bar{c}^T \bar{x}$.
- ▶ One iteration seeks vertex \bar{v}_{i+1} "adjacent" to \bar{v}_i s.t. $\bar{c}^T \bar{v}_{i+1} > \bar{c}^T \bar{v}_i$.
- ▶ For $i = 1$, the initial vertex \bar{v}_1 of M_F is $\bar{x} = \bar{0}$, $\bar{z} = \bar{0}$

Example (cont):

$$\bar{v}_1 = [x_1 \ x_2 \ y_1 \ y_2 \ z]^T = [0 \ 0 \ 0 \ 0 \ 0]^T$$

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Find \bar{u}

Construct vector \bar{u} s.t.

$$\bar{u}^T A = \bar{c}^T \quad (7.4)$$

- ▶ Given \bar{v}_i
- ▶ Construct $n \times n$ nonsingular submatrix A_i with corresponding rows \bar{b}_i s.t.

$$A_i \bar{v}_i = \bar{b}_i$$

- ▶ Let $R =$ rows of A in A_i
- ▶ Solve

$$A_i^T \bar{u}_i = \bar{c}$$

- ▶ Let \bar{u} be \bar{u}_i for indices in R and 0's for indices not in R .

Example (cont)

Choose the first five rows of A and \bar{b} ($R = [1; 2; 3; 4; 5]$) since

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{v}_1} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{b}_1}$$

Solving (Gaussian elimination)

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}}_{A_1^T} \bar{u}_1 = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \end{bmatrix}}_{\bar{c}}$$

yields

$$\bar{u}_1^T = [-1 \ 1 \ -1 \ 1 \ 1]$$

Then

$$\bar{u} = [-1 \ 1 \ -1 \ 1 \ 1 \ 0 \ 0]^T$$

Case 1: $\bar{u} \geq \bar{0}$

In this case, \bar{v}_i is actually the optimal point with optimal value $\bar{c}^T \bar{v}_i$.

Case 2: $\bar{u} \not\geq \bar{0}$, i.e. there exists some $u_k < 0$

In this case, \bar{v}_i is not the optimal point. We need to move along an edge to an adjacent vertex to increase the value of the objective function.

- ▶ Let k be the lowest index of \bar{u} s.t. $u_k < 0$ (must be $k \in R$)
- ▶ Let k' be the corresponding row of \bar{u}_i and A_i and the corresponding column of $-A_i^{-1}$

Find \bar{y}

- ▶ Let \bar{y} be the k' th column of $-A_i^{-1}$. Solve

$$A_i \bar{y} = -e_{k'}$$

Then

$$\bar{a}_\ell \bar{y} = 0 \quad \text{for every row } \bar{a}_\ell \text{ of } A_i, \ell \neq k'$$

$$\bar{a}_{k'} \bar{y} = -1 \quad \text{for the } k' \text{th row } \bar{a}_{k'} \text{ of } A_i$$

The vector \bar{y} provides the direction along which to move to the next vertex.

Example:

We found so far

$$\bar{u}_1 = [-1 \ 1 \ -1 \ 1 \ 1]^T \text{ and } \bar{u} = [-1 \ 1 \ -1 \ 1 \ 1 \ 0 \ 0]^T$$

$k = 1$ since the first row of \bar{u} is -1 . $k' = 1$ since it is also the first row of \bar{u}_i .

Thus, solve

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}}_{A_1} \bar{y} = \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{-e_1}$$

for \bar{y} , yielding $\bar{y} = [1 \ 0 \ 0 \ 0 \ 0]^T$

Find λ and v_{i+1}

We move along edge \bar{y} to better vertex \bar{v}_{i+1} .

- ▶ Let $S =$ indices ℓ s.t. $\bar{a}_\ell \bar{y} > 0$
- ▶ Find greatest $\lambda \geq 0$ s.t.

$$A(\bar{v}_i + \lambda \bar{y}) \leq \bar{b}$$

Choose λ_i s.t.

$$\begin{aligned} \bar{a}_\ell(\bar{v}_i + \lambda_i \bar{y}) &= b_\ell \quad \text{for some } \ell \in S \\ \bar{a}_m(\bar{v}_i + \lambda_i \bar{y}) &\leq b_m \quad \text{for other } m \neq \ell \end{aligned}$$

- ▶ Set

$$\boxed{\bar{v}_{i+1} = \bar{v}_i + \lambda_i \bar{y}} \quad (7.7)$$

Vertex \bar{v}_{i+1} is discovered by moving along ray \bar{y} as far as possible without violating the constraints. Moreover

$$\bar{c}^T \bar{v}_{i+1} > \bar{c}^T \bar{v}_i$$

- ▶ Construct A_{i+1} from A_i for next iteration by substituting row \bar{a}_ℓ of A for row $\bar{a}_{\ell'}$ of A_i

Since there are only finite number of vertices to examine, Case 1 eventually occurs.

Final Step: Satisfiability

The $T_{\mathbb{Q}}$ -formula F (7.1) is satisfiable

iff

the returned optimal value of the optimization problem M_F (7.2) is $\bar{1}^T \bar{e}_2$.

Example (cont): We found in Step 1

$$\bar{y} = [1 \ 0 \ 0 \ 0 \ 0]^T$$

where

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{y}} = \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{-\bar{e}_1}$$

Example (cont)

Compute $A\bar{y}$

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & -1 & -1 \end{bmatrix}}_A \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{y}} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$S = [7]$ since $\bar{a}_7 \bar{y} = 1 > 0$. Examining the 7th row of the constraints, choose the greatest λ_1 such that (7.7b)

$$\underbrace{[1 \ -1 \ 1 \ -1 \ -1]}_{\bar{a}_7} (\bar{v}_1 + \lambda_1 \bar{y}) = [1 \ -1 \ 1 \ -1 \ -1] \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) = 1$$

that is, choose $\lambda_1 = 1$. Therefore, (7.7c)

$$\boxed{\bar{v}_2 = \bar{v}_1 + \lambda_1 \bar{y} = [1 \ 0 \ 0 \ 0 \ 0]^T}$$

Example (cont):

Form A_2 from A_1 replacing the 1st row ($k' = 1$) of A_1 by the 7th row ($\ell = 7$) of A .

$$A_2 = \begin{bmatrix} 1 & -1 & 1 & -1 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

This move to vertex \bar{v}_2 makes progress:

$$\underbrace{[1 \ -1 \ 1 \ -1 \ -1]}_{\bar{c}^T} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{v}_1} = 0 < \underbrace{[1 \ -1 \ 1 \ -1 \ -1]}_{\bar{c}^T} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{v}_2} = 1.$$

Now $R = [7; 2; 3; 4; 5]$ (rows of A in A_2).

Example (cont):

Solve

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 \end{bmatrix}}_{A_2^T} \bar{u}_2 = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \end{bmatrix}}_{\bar{c}}$$

for \bar{u}_2 yielding $\bar{u}_2 = [1 \ 0 \ 0 \ 0 \ 0]^T$. Since $\bar{u}_2 \geq 0$, we are in Case 1: we have found an optimum point, \bar{v}_2 , with optimal value 1.

Since we have that $v_F = \bar{I}^T \bar{e}_2 = 1$, the equality of the optimal point and v_F implies that

$$F: x + y \geq 1 \wedge x - y \geq -1$$

is $T_{\mathbb{Q}}$ -satisfiable. In particular, extract from

$$\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z \end{bmatrix} = \bar{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

the assignment

$$x = x_1 - x_2 = 1 - 0 = 1 \quad \text{and} \quad y = y_1 - y_2 = 0 - 0 = 0$$

Example (cont)

Then (7.2):

$$\begin{aligned} \max \quad & \bar{I}^T (D_2 \bar{x} - \bar{z}) \\ \text{subject to} \quad & \bar{x}, \bar{z} \geq \bar{0} \\ & D_1 \bar{x} \leq \bar{e}_1 \\ & D_2 \bar{x} - \bar{z} \leq \bar{e}_2 \end{aligned}$$

Expanding, we have

$$\begin{aligned} \bar{c}^T \bar{x} &= \bar{I}^T [D_2 \quad -I] \begin{bmatrix} x \\ y \\ z_1 \\ z_2 \end{bmatrix} = [1 \ 1] \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z_1 \\ z_2 \end{bmatrix} \\ &= \underbrace{[1 \ 1 \ -1 \ -1]}_{\bar{c}^T} \begin{bmatrix} x \\ y \\ z_1 \\ z_2 \end{bmatrix} \end{aligned}$$

Example: $\Sigma_{\mathbb{Q}}$ -formula (7.1)

$$F: x \geq 0 \wedge y \geq 0 \wedge x \geq 2 \wedge y \geq 2 \wedge x + y \leq 3,$$

or, in matrix form,

$$F: \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ -2 \\ -2 \\ 3 \end{bmatrix}$$

Is F $T_{\mathbb{Q}}$ -satisfiable?

Step 0

Because x and y are already constrained to be nonnegative, we do not need to introduce new x_1, x_2, y_1, y_2 . Rewrite:

$$\underbrace{[1 \ 1]}_{D_1} \begin{bmatrix} x \\ y \end{bmatrix} \leq \underbrace{[3]}_{\bar{e}_1} \quad \text{and} \quad \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{D_2} \begin{bmatrix} x \\ y \end{bmatrix} \geq \underbrace{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}_{\bar{e}_2}$$

so that $\bar{e}_1 \geq 0$ and $\bar{e}_2 > 0$.

Example (cont):

obtaining the optimization problem (7.3)

$$\begin{aligned} \max \quad & \underbrace{[1 \ 1 \ -1 \ -1]}_{\bar{c}^T} \begin{bmatrix} x \\ y \\ z_1 \\ z_2 \end{bmatrix} \\ \text{subject to} \quad & \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}}_A \begin{bmatrix} x \\ y \\ z_1 \\ z_2 \end{bmatrix} \leq \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 2 \\ 2 \end{bmatrix}}_{\bar{b}} \end{aligned}$$

Example (cont):

Use the initial vertex

$$\bar{v}_1 = \begin{bmatrix} x \\ y \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

in **Step 1**.

F is satisfiable iff the optimal value v_F is equal to

$$\bar{1}^T \bar{e}_2 = [1 \ 1] \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 4.$$

We use the simplex algorithm to find the optimum.

Example (cont):

Since $u_1, u_2 < 0$, we are in Case 2 with $k = k' = 1$. Let \bar{y} be the first column of $-A_1^{-1}$: solve

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{A_1} \bar{y} = \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{-\bar{e}_1}$$

to yield $\bar{y} = [1 \ 0 \ 0 \ 0]^T$. Then $S = [5; 6]$; i.e., the 5th and 6th rows \bar{a} of A are such that $\bar{a}\bar{y} > 0$. Choose the largest λ_1 such that

$$A(\bar{v}_1 + \lambda_1 \bar{y}) \leq \bar{b}$$

Example (cont):

Step 1

Choose rows $R = [1; 2; 3; 4]$ of A and \bar{b} , giving

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{v}_1} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{b}_1}$$

Solving

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{A_1^T} \bar{u}_1 = \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}}_{\bar{c}}$$

yields $\bar{u}_1 = [-1 \ -1 \ 1 \ 1]^T$. Adding 0s for the rows not in R produces \bar{u} :

$$\bar{u} = [-1 \ -1 \ 1 \ 1 \ 0 \ 0 \ 0]^T$$

Example (cont):

Focusing on the 5th and 6th rows of A (since $S' = [5; 6]$), choose the largest λ_1 such that

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}}_{\text{rows 5,6 of } A} \left(\underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{v}_1} + \lambda_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{y}} \right) \leq \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{\text{rows 5,6 of } \bar{b}}$$

Namely, choose $\lambda_1 = 2$ (and $\ell = 6$). Then

$$\bar{v}_2 = \bar{v}_1 + \lambda_1 \bar{y} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example (cont):

Replace the 1st row of A_1 (since $k' = 1$) by the 6th row of A (since $\ell = 6$) to produce

$$A_2 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \bar{b}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Have we made progress? Yes, for

$$\bar{c}^T \bar{v}_1 = 0 < 2 = \bar{c}^T \bar{v}_2 .$$

The objective function has increased from 0 to 2.

Example (cont):

Step 2

Now $R = [6; 2; 3; 4]$ (the indices of rows of A in A_2). Solve

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{A_2^T} \bar{u}_2 = \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}}_{\bar{c}}$$

to yield $\bar{u}_2 = [\frac{1}{6} \quad -\frac{1}{2} \quad 0 \quad \frac{1}{3} \quad \frac{1}{4}]^T$. Then filling in 0s for the other rows of A produces:

$$\bar{u} = [0 \quad -\frac{1}{2} \quad 0 \quad \frac{1}{3} \quad 0 \quad \frac{1}{6} \quad 0]^T$$

$u_2 < 0$, so $k = 2$, which corresponds to row $k' = 2$ of \bar{u}_2 .

According to Case 2, let \bar{y} be the 2nd column of $-A_2^{-1}$: solve $A_2 \bar{y} = -e_2$ to yield $\bar{y} = [0 \ 1 \ 0 \ 0]^T$. Then the 5th and 7th rows \bar{a} of A are such that $\bar{a}\bar{y} > 0$ so that $S = [5; 7]$.

Example (cont):

Focusing on the 5th and 7th rows of A , choose the largest λ_2 such that

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}}_{\text{rows 5,7 of } A} \left(\underbrace{\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{v}_2} + \lambda_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\bar{b}} \right) \leq \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{\text{rows 5,7 of } \bar{b}}$$

Choose $\lambda_2 = 1$ (and $\ell = 5$). Then

$$\bar{v}_3 = \bar{v}_2 + \lambda_2 \bar{y} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Example (cont):

Replace the 2nd row of A_2 (since $k' = 2$) by the 5th row of A (since $\ell = 5$) to produce

$$A_3 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \bar{b}_3 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

Have we made progress? Yes, for

$$\bar{c}^T \bar{v}_1 = 0 < \bar{c}^T \bar{v}_2 = 2 < \bar{c}^T \bar{v}_3 = 3 .$$

The objective function has increased from 2 to 3.

Example (cont):

Step 3

Now $R = [6; 5; 3; 4]$. Solve $A_3^T \bar{u}_3 = \bar{c}$, yielding $\bar{u}_3 = [0 \ 1 \ 1 \ 1]^T$.

Now $\bar{u}_3 \geq \bar{0}$, so we are in Case 1: \bar{v}_3 is the optimum with objective value

$$\underbrace{[1 \ 1 \ -1 \ -1]}_{\bar{c}^T} \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\bar{v}_3} = 3 .$$

Final Step: Satisfiability

The optimal value of the constructed optimization problem is 3, which is less than the required $v_F = 4$ of **Step 0**. Hence, F is $T_{\mathbb{Q}}$ -unsatisfiable.