8. Quantifier-Free Linear Arithmetic

Decision Procedures for Quantifier-free Fragments

For theory T with signature Σ and axioms Σ-formulae of form

$$\forall x_1, \ldots, x_n.\ F[x_1, \ldots, x_n]$$

Decide if

$$F[x_1, \ldots, x_n] \text{ or } \exists x_1, \ldots, x_n.\ F[x_1, \ldots, x_n] \text{ is } T\text{-satisfiable}$$

$$\left[\begin{array}{l} \text{Decide if} \\ \qquad F[x_1, \ldots, x_n] \text{ or } \forall x_1, \ldots, x_n.\ F[x_1, \ldots, x_n] \text{ is } T\text{-valid} \end{array}\right]$$

where F is quantifier-free and $\text{free}(F) = \{x_1, \ldots, x_n\}$

Note: no quantifier alternations

We consider only conjunctive quantifier-free Σ-formulae, i.e., conjunctions of Σ-literals (Σ-atoms or negations of Σ-atoms).

For given arbitrary quantifier-free Σ-formula F, convert it into DNF Σ-formula

$$F_1 \ \vee \ \ldots \ \vee \ F_k$$

where each F_i conjunctive.

F is T-satisfiable iff at least one F_i is T-satisfiable.

Preliminary Concepts

Vector

variable n-vector

$$\overline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

n-vector $\overline{a} \in \mathbb{Q}^n$

$$\overline{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

transpose

$$\overline{a}^{\mathsf{T}} = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$$

Matrix

$m \times n$-matrix

$A \in \mathbb{Q}^{m \times n}$

$$A = \begin{bmatrix} a_{11} \cdots a_{1n} \\ \vdots \\ a_{m1} \cdots a_{mn} \end{bmatrix}$$

transpose

$$A^{\mathsf{T}} = \begin{bmatrix} a_{11} \cdots a_{m1} \\ \vdots \\ a_{1n} \cdots a_{mn} \end{bmatrix}$$

column

row

$$\begin{bmatrix} & a_{1j} & \\ & \vdots & \\ a_{i1} \cdots & a_{ij} & \cdots a_{in} \\ & \vdots & \\ & a_{mj} & \end{bmatrix}$$

Multiplication

vector-vector

$$\overline{a}^{\mathsf{T}}\overline{b} = [a_1 \;\; \cdots \;\; a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^{n} a_i b_i$$

matrix-vector

$$A\overline{x} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{1i}x_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi}x_i \end{bmatrix}$$

matrix-matrix

$$\underset{A}{\begin{bmatrix} & & \\ & a_{ik} & \\ & & \end{bmatrix}} \underset{B}{\begin{bmatrix} & & \\ & b_{kj} & \\ & & \end{bmatrix}} = \underset{P}{\begin{bmatrix} & & \\ & p_{ij} & \\ & & \end{bmatrix}}$$

where $p_{ij} = \overline{a}_i \overline{b}_j = \begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix} = \sum_{k=1}^{n} a_{ik} b_{kj}$

Special Vectors and Matrices

$\overline{0}$ - vector (column) of 0s

$\overline{1}$ - vector of 1s

Thus $\overline{1}^{\mathsf{T}}\overline{x} = \sum_{i=1}^{n} x_i$

$$I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$ identity matrix ($n \times n$)

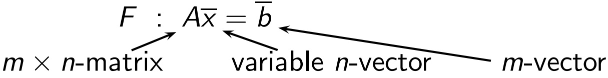
Thus $IA = AI = A$

unit vector $e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \!\!\leftarrow i$

Vector Space - set S of vectors closed under addition and scaling of vectors. That is,

$$\text{if } \overline{v}_1, \dots, \overline{v}_k \in S \quad \text{then} \quad \lambda_1 \overline{v}_1 + \cdots + \lambda_k \overline{v}_k \in S$$
$$\text{for } \lambda_1, \dots, \lambda_n \in \mathbb{R}.$$

Linear Equation



$$F \ : \ A\overline{x} = \overline{b}$$

$m \times n$-matrix variable n-vector m-vector

represents the $\Sigma_{\mathbb{Q}}$-formula

$$F : \ (a_{11}x_1 + \cdots + a_{1n}x_n = b_1) \wedge \cdots \wedge (a_{m1}x_1 + \cdots + a_{mn}x_n = b_m)$$

Gaussian Elimination

Find $\overline{x}$ s.t. $A\overline{x} = \overline{b}$ by elementary row operations

- Swap two rows.
- Multiply a row by a nonzero scalar.
- Add one row to another.

Example:

Solve

$$\begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix}$$

Construct the augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 6 \\ 1 & 0 & 1 & 1 \\ 2 & 2 & 1 & 2 \end{array}\right]$$

Apply the row operations as follows:

1. Add $-2\overline{a}_1 + 4\overline{a}_2$ to $\overline{a}_3$

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 6 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -6 \end{array}\right]$$

2. Add $-\overline{a}_1 + 2\overline{a}_2$ to $\overline{a}_2$

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 6 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 1 & -6 \end{array}\right]$$

This augmented matrix is in $\underline{\text{triangular}}$ form.

Solving

$$\begin{aligned} &x_3 = -6 \\ &-x_2 - x_3 = -3 \quad \Rightarrow \quad x_2 = -3 \\ &3x_1 + x_2 + 2x_3 = 6 \quad \Rightarrow \quad x_1 = 7 \end{aligned}$$

The solution is $\overline{x} = [7 \ \ -3 \ \ -6]^{\mathsf{T}}$

Inverse Matrix

A^{-1} is the inverse matrix of square matrix A if

$$AA^{-1} = A^{-1}A = I$$

Square matrix A is nonsingular (invertible) if its inverse A^{-1} exists.

How to compute A^{-1} of A?

$$[A \mid I] \xrightarrow[\substack{\text{elementary}\\ \text{row operations}}]{} [I \mid A^{-1}]$$

How to compute kth column of A^{-1}?
Solve $A\overline{y} = e_k$, i.e.

$$\left[\begin{array}{ccc|c} & & & 0 \\ & & & \vdots \\ & A & & 1 \\ & & & \vdots \\ & & & 0 \end{array}\right] \xrightarrow[\substack{\text{elementary}\\ \text{row operations}}]{} \begin{array}{c} \overline{y} = \dots \\ (k\text{th column of } A^{-1}) \end{array}$$

Linear Inequality

$$G: \; A\overline{x} \leq b$$

represents the $\Sigma_{\mathbb{Q}}$-formula

$$G: \; (a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1) \wedge \cdots \wedge (a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m)$$

The inequality describes a polyhedron in $\mathbb{R}^n$.

For $m \times n$-matrix A, m-vector b, variable n-vector $\overline{x}$ where $m \geq n$:

An n-vector $\overline{v}$ is a vertex of $A\overline{x} \leq b$ if there is nonsingular $n \times n$-submatrix A_0 and corresponding n-subvector b_0 s.t.

$$A_0\overline{v} = b_0$$

Optimization Problem

$$\begin{array}{llll} \textbf{max} & \overline{c}^{\mathsf{T}}\overline{x} & & \ldots\text{objective function} \\ \textbf{subject to} & & & \\ & A\overline{x} \leq \overline{b} & & \ldots\text{constraints} \end{array}$$

Solution: vertex $\overline{v}^*$ satisfying $A\overline{x} \leq \overline{b}$ and maximize $\overline{c}^{\mathsf{T}}\overline{x}$. That is,

$A\overline{v}^* \leq \overline{b}$ and
$\overline{c}^{\mathsf{T}}\overline{v}^*$ is maximal: $\overline{c}^{\mathsf{T}}\overline{v}^* \geq \overline{c}^{\mathsf{T}}\overline{u}$ for all $\overline{u}$ satisfying $A\overline{u} \leq \overline{b}$

- If $A\overline{x} \leq \overline{b}$ is unsatisfiable $\Rightarrow$ maximum is $-\infty$
- It's possible that the maximum is unbounded
  $\Rightarrow$ maximum is ∞

Example: Consider optimization problem:

$$\textbf{max} \quad \underbrace{\begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}}_{\overline{c}^{\mathsf{T}}} \underbrace{\begin{bmatrix} x \\ y \\ z_1 \\ z_2 \end{bmatrix}}_{\overline{x}}$$

subject to

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z_1 \\ z_2 \end{bmatrix}}_{\overline{x}} \leq \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 2 \\ 2 \end{bmatrix}}_{\overline{b}}$$

A is a 7×4-matrix, $\overline{b}$ is a 7-vector, and
$\overline{x}$ is a variable 4-vector representing the four variables $\{x, y, z_1, z_2\}$.

Example (cont):

The objective function is

$$(x - z_1) + (y - z_2) \ .$$

The constraints are equivalent to the $\Sigma_{\mathbb{Q}}$-formula

$$\begin{array}{l} x \geq 0 \ \wedge \ y \geq 0 \ \wedge \ z_1 \geq 0 \ \wedge \ z_2 \geq 0 \\ \ \wedge \ x + y \leq 3 \ \wedge \ x - z_1 \leq 2 \ \wedge \ y - z_2 \leq 2 \end{array}$$

$\overline{v} = [2 \ 1 \ 0 \ 0]^{\mathsf{T}}$ is a <u>vertex</u> of the constraints. For the nonsingular submatrix A_0 (rows 3, 4, 5, 6 of A), we have

$$\underbrace{\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}}_{A_0} \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\overline{v}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 3 \\ 2 \end{bmatrix}}_{b_0}$$

Duality Theorem

For $A \in \mathbb{Z}^{m\times n}$, $\overline{b} \in \mathbb{Z}^{m}$, $\overline{c} \in \mathbb{Z}^{n}$,

$$\max\{\overline{c}^{\mathsf{T}}\overline{x} \mid A\overline{x} \leq \overline{b}\} = \min\{\overline{y}^{\mathsf{T}}\overline{b} \mid \overline{y} \geq \overline{0} \ \wedge \ \overline{y}^{\mathsf{T}}A = \overline{c}^{\mathsf{T}}\}$$

if the constraints are satisfiable.

That is,

maximizing the function $c^{\mathsf{T}}\overline{x}$ over $A\overline{x} \leq \overline{b}$
(the primal form of the optimization problem)

is equivalent to

minimizing the function $\overline{y}^{\mathsf{T}}\overline{b}$ over all the nonnegative $\overline{y}$
s.t. $\overline{y}^{\mathsf{T}}A = \overline{c}^{\mathsf{T}}$
(the dual form of the optimization problem)

Outline of Algorithm

Given $\Sigma_{\mathbb{Q}}$-formula

$$F: \; a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1 \;\wedge\; \cdots \;\wedge\; a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m \tag{7.1}$$

or in matrix notation

$$F: \; A\overline{x} \leq \overline{b}$$

Note: • equations

$$a_{i1}x_1 + \ldots + a_{in}x_n = b_i$$

are allowed — break into two inequalities

$$a_{i1}x_1 + \ldots + a_{in}x_n \leq b_i \;\wedge\; -a_{i1}x_1 - \ldots - a_{in}x_n \leq -b_i .$$

• Strict inequalities

$$a_{i1}x_1 + \cdots + a_{in}x_n < b_i \; .$$

excluded from our discussion — but can be added.

Outline of Algorithm (cont)

To determine the satisfiability of F,

Step 0: reformulate the satisfiability of F as an optimization problem

$$M_F : \max\{\overline{c}^{\mathsf{T}}\overline{x}' \mid A'\overline{x}' \leq \overline{b}'\}$$

s.t. F is $T_{\mathbb{Q}}$-satisfiable iff the optimal value of M_F is a particular value v_F (derived from the structure of F)

Step 1, Step 2, ... (until termination) execute the simplex method

Outline of Algorithm (cont)

The simplex method traverses the vertices of $A'\overline{x}' \leq \overline{b}'$ searching for the maximum of the objective function $\overline{c}^{\mathsf{T}}\overline{x}'$:
if $\overline{v}_1, \overline{v}_2, \ldots$ are the traversed vertices in Step 1, Step 2, …, then

$$\overline{c}^{\mathsf{T}}\overline{v}_1 < \overline{c}^{\mathsf{T}}\overline{v}_2 < \cdots \ .$$

The simplex method terminates at some vertex $\overline{v}_{i*}$ where $\overline{c}^{\mathsf{T}}\overline{v}_{i*}$ is the global optimum

Final step: Compare the discovered optimal value $\overline{c}^{\mathsf{T}}\overline{v}_{i*}$ to the desired value v_F.

- if equal, then F is $T_{\mathbb{Q}}$-satisfiable
- otherwise, F is $T_{\mathbb{Q}}$-unsatisfiable

Step 0: From Satisfiability to Optimization

Given $\Sigma_{\mathbb{Q}}$-formula

$$F : \ A\overline{x} \leq \overline{b} \tag{7.1}$$

reformulate to new constraint system (new A, $\overline{x}$, $\overline{b}$)

$$F' : \ \overline{x} \geq 0, \ A\overline{x} \leq \overline{b}$$

F' is $T_{\mathbb{Q}}$-equisatisfiable to F

The trick: replace each variable x in F by $x_1 - x_2$

Separate the new constraints $A\overline{x} \leq \overline{b}$ into

$$D_1\overline{x} \leq \overline{e}_1 \quad \text{and} \quad D_2\overline{x} \geq \overline{e}_2 \quad \text{for } \overline{e}_1 \geq \overline{0},\ \overline{e}_2 > \overline{0}$$

Example: $\Sigma_{\mathbb{Q}}$-formula

$$F: \; x + y \geq 1 \;\wedge\; x - y \geq -1 \;.$$

To convert it to the form $\overline{x} \geq \overline{0} \;\wedge\; A\overline{x} \leq \overline{b}$, introduce nonnegative x_1, x_2 for x and y_1, y_2 for y:

$$F': \begin{array}{l} (x_1 - x_2) + (y_1 - y_2) \geq 1 \;\wedge\; (x_1 - x_2) - (y_1 - y_2) \geq -1 \\ \quad\wedge\; x_1, x_2, y_1, y_2 \geq 0 \end{array}$$

F is $T_{\mathbb{Q}}$-equisatisfiable to F'. In matrix form,

$$F': \; \underbrace{\begin{bmatrix} -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} \leq \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\overline{b}}$$

Since $b_1 < 0$ and $b_2 > 0$, separating constraints yields

$$\underbrace{\begin{bmatrix} -1 & 1 & 1 & -1 \end{bmatrix}}_{D_1} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} \leq \underbrace{[1]}_{\overline{e}_1} \text{ and } \underbrace{\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}}_{D_2} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} \geq \underbrace{[1]}_{\overline{e}_2}$$

Generate the optimization problem:

$$M_F : \ \textbf{max} \quad \overline{1}^{\mathsf{T}}(D_2\overline{x} - \overline{z}) \tag{7.2}$$

$$\begin{array}{lrcll} \textbf{subject to} & & & & \\ & \overline{x}, \overline{z} & \geq & \overline{0} & \qquad (1) \\ & D_1\overline{x} & \leq & \overline{e}_1 & \qquad (2) \\ & D_2\overline{x} - \overline{z} & \leq & \overline{e}_2 & \qquad (3) \end{array}$$

length of variable vector $\overline{z} = \#$ of rows of D_2

- The point $\overline{x} = \overline{0}$, $\overline{z} = \overline{0}$ satisfies constraints (1) – (3). It's a vertex.
- The optimum v_F equals $\overline{1}^{\mathsf{T}}\overline{e}_2$ iff F is $T_{\mathbb{Q}}$-satisfiable.

M_F can be written in standard form as

$$M_F : \textbf{max} \quad \underbrace{\overline{1}^{\mathsf{T}} \begin{bmatrix} D_2 & -I \end{bmatrix}}_{\overline{c}^{\mathsf{T}}} \underbrace{\begin{bmatrix} \overline{x} \\ \overline{z} \end{bmatrix}}_{\overline{y}} \tag{7.3}$$

subject to

$$\underbrace{\begin{bmatrix} -I & \\ & -I \\ D_1 & \\ D_2 & -I \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \overline{x} \\ \overline{z} \end{bmatrix}}_{\overline{y}} \leq \underbrace{\begin{bmatrix} \overline{0} \\ \overline{0} \\ \overline{e}_1 \\ \overline{e}_2 \end{bmatrix}}_{\overline{b}}$$

Example (cont):

$$\underbrace{\left[\begin{array}{cccc} -1 & 1 & 1 & -1 \end{array}\right]}_{D_1} \left[\begin{array}{c} x_1 \\ x_2 \\ y_1 \\ y_2 \end{array}\right] \leq \underbrace{[1]}_{\overline{e}_1} \text{ and } \underbrace{\left[\begin{array}{cccc} 1 & -1 & 1 & -1 \end{array}\right]}_{D_2} \left[\begin{array}{c} x_1 \\ x_2 \\ y_1 \\ y_2 \end{array}\right] \geq \underbrace{[1]}_{\overline{e}_2}$$

D_2 has only one row, so $\overline{z} = [z]$. Pose the following optimization problem:

$$\begin{array}{ll} \textbf{max} & \left[\begin{array}{cccc} 1 & -1 & 1 & -1 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ y_1 \\ y_2 \end{array}\right] - [z] \\ \textbf{subject to} & \\ & x_1, x_2, y_1, y_2, z \geq 0 \\ & \left[\begin{array}{cccc} -1 & 1 & 1 & -1 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ y_1 \\ y_2 \end{array}\right] \leq [1] \\ & \left[\begin{array}{cccc} 1 & -1 & 1 & -1 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ y_1 \\ y_2 \end{array}\right] - [z] \leq [1] \end{array}$$

F is $T_{\mathbb{Q}}$-satisfiable iff the optimum is $\overline{1}^{\mathsf{T}}\overline{e}_2 = 1$.

$[x_1 \ x_2 \ y_1 \ y_2 \ z] = [0 \ 0 \ 0 \ 0 \ 0]$ is a vertex.

Example (cont):

Rewriting the optimization problem

$$\textbf{max} \quad \underbrace{\begin{bmatrix} 1 & -1 & 1 & -1 & -1 \end{bmatrix}}_{\bar{c}^{\mathsf{T}}} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z \end{bmatrix}$$

subject to

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & -1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z \end{bmatrix} \leq \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{\bar{b}}$$

Step i ($i \geq 1$): Find a Better Vertex

Optimization problem of form

$$\begin{array}{lrcl} \textbf{max} & \overline{c}^{\mathsf{T}}\overline{x} & & \\ \textbf{subject to} & & & \\ & A\overline{x} & \leq & \overline{b} \end{array}$$

we are given satisfying vertex $\overline{v}_i$.

- The simplex method traverses vertices of the space defined by $A\overline{x} \leq \overline{b}$ to find the vertex $\overline{v}^*$ that maximizes $\overline{c}^{\mathsf{T}}\overline{x}$.
- One iteration seeks vertex $\overline{v}_{i+1}$ "adjacent" to $\overline{v}_i$ s.t. $\overline{c}^{\mathsf{T}}\overline{v}_{i+1} > \overline{c}^{\mathsf{T}}\overline{v}_i$.
- For $i = 1$, the initial vertex $\overline{v}_1$ of M_F is $\overline{x} = \overline{0}$, $\overline{z} = \overline{0}$

Example (cont):

$$\overline{v}_1 = [x_1 \ x_2 \ y_1 \ y_2 \ z]^{\mathsf{T}} = [0 \ 0 \ 0 \ 0 \ 0]^{\mathsf{T}}$$

<u>Find $\overline{u}$</u>

Construct vector $\overline{u}$ s.t.

$$\overline{u}^{\mathsf{T}} A = \overline{c}^{\mathsf{T}} \tag{7.4}$$

- Given $\overline{v}_i$
- Construct $n \times n$ nonsingular submatrix A_i with corresponding rows $\overline{b}_i$ s.t.

  $$A_i \overline{v}_i = \overline{b}_i$$

- Let $R =$ rows of A in A_i
- Solve

  $$A_i{}^{\mathsf{T}} \overline{u}_i = \overline{c}$$

- Let $\overline{u}$ be $\overline{u}_i$ for indices in R and
  0's for indices not in R.

Example (cont)

Choose the first five rows of A and $\overline{b}$ ($R = [1;2;3;4;5]$) since

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\overline{v}_1} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\overline{b}_1}$$

Solving (Gaussian elimination)

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}}_{{A_1}^{\mathsf{T}}} \overline{u}_1 = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \end{bmatrix}}_{\overline{c}}$$

yields

$${\overline{u}_i}^{\mathsf{T}} = \begin{bmatrix} -1 & 1 & -1 & 1 & 1 \end{bmatrix} .$$

Then

$$\overline{u} = \begin{bmatrix} -1 & 1 & -1 & 1 & 1 & 0 & 0 \end{bmatrix}^{\mathsf{T}}$$

Case 1: $\overline{u} \geq \overline{0}$

In this case, $\overline{v}_i$ is actually the optimal point with optimal value $\overline{c}^\mathsf{T}\overline{v}_i$.

Case 2: $\overline{u} \not\geq \overline{0}$, i.e. there exists some $u_k < 0$

In this case, $\overline{v}_i$ is not the optimal point. We need to move along an edge to an adjacent vertex to increase the value of the objective function.

- Let k be the lowest index of $\overline{u}$ s.t. $u_k < 0$ (must be $k \in R$)
- Let k' be the corresponding row of $\overline{u}_i$ and A_i and the corresponding column of $-A_i^{-1}$

Find $\overline{y}$

- Let $\overline{y}$ be the k'th column of $-A_i^{-1}$. Solve

  $$A_i\overline{y} = -\mathrm{e}_{k'}$$

  Then

  $$\begin{array}{ll} \overline{a}_\ell\overline{y} = 0 & \text{for every row } \overline{a}_\ell \text{ of } A_i,\ \ell \neq k' \\ \overline{a}_{k'}\overline{y} = -1 & \text{for the } k'\text{th row } \overline{a}_{k'} \text{ of } A_i \end{array}$$

  The vector $\overline{y}$ provides the direction along which to move to the next vertex.

Example:

We found so far

$$\overline{u}_1 = \begin{bmatrix} -1 & 1 & -1 & 1 & 1 \end{bmatrix}^\mathsf{T} \text{ and } \overline{u} = \begin{bmatrix} -1 & 1 & -1 & 1 & 1 & 0 & 0 \end{bmatrix}^\mathsf{T}$$

$k = 1$ since the first row of $\overline{u}$ is -1. $k' = 1$ since it is also the first row of $\overline{u}_i$.

Thus, solve

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}}_{A_1} \overline{y} = \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{-\mathsf{e}_1}$$

for $\overline{y}$, yielding $\overline{y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^\mathsf{T}$

Find λ and v_{i+1}

We move along edge $\overline{y}$ to better vertex $\overline{v}_{i+1}$.

- Let $S =$ indices ℓ s.t. $\overline{a}_\ell \overline{y} > 0$
- Find greatest $\lambda \geq 0$ s.t.

  $$A(\overline{v}_i + \lambda \overline{y}) \leq \overline{b}$$

  Choose λ_i s.t.

  $$\begin{array}{ll} \overline{a}_\ell(\overline{v}_i + \lambda_i \overline{y}) = b_\ell & \text{for some } \ell \in S \\ \overline{a}_m(\overline{v}_i + \lambda_i \overline{y}) \leq b_m & \text{for other } m \neq \ell \end{array}$$

- Set

  $$\boxed{\overline{v}_{i+1} = \overline{v}_i + \lambda_i \overline{y}} \qquad (7.7)$$

  Vertex $\overline{v}_{i+1}$ is discovered by moving along ray $\overline{y}$ as fas as possible without violating the constraints. Moreover

  $$\overline{c}^{\mathsf{T}} \overline{v}_{i+1} > \overline{c}^{\mathsf{T}} \overline{v}_i$$

- Construct A_{i+1} from A_i for next iteration by substituting row $\overline{a}_\ell$ of A for row $\overline{a}_{k'}$ of A_i

Since there are only finite number of vertices to examine, <u>Case 1</u> eventually occurs.

Final Step: Satisfiability

The $T_{\mathbb{Q}}$-formula F (7.1) is satisfiable

iff

the returned optimal value of the optimization problem M_F (7.2) is $\overline{1}^{\mathsf{T}}\overline{e}_2$.

Example (cont): We found in Step 1

$$\overline{y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathsf{T}}$$

where

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\overline{y}} = \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{-\mathrm{e}_1}$$

Example (cont)

Compute $A\overline{y}$

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & -1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\overline{y}} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$S = [7]$ since $\overline{a}_7\overline{y} = 1 > 0$. Examining the 7th row of the constraints, choose the greatest λ_1 such that (7.7b)

$$\underbrace{\begin{bmatrix} 1 & -1 & 1 & -1 & -1 \end{bmatrix}}_{\overline{a}_7}(\overline{v}_1+\lambda_1\overline{y}) = \begin{bmatrix} 1 & -1 & 1 & -1 & -1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) = 1$$

that is, choose $\lambda_1 = 1$. Therefore, (7.7c)

$$\boxed{\overline{v}_2 = \overline{v}_1 + \lambda_1\overline{y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T}$$

Example (cont):

Form A_2 from A_1 replacing the 1st row ($k' = 1$) of A_1 by the 7th row ($\ell = 7$) of A.

$$A_2 = \begin{bmatrix} 1 & -1 & 1 & -1 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

This move to vertex $\overline{v}_2$ makes progress:

$$\underbrace{\begin{bmatrix} 1 & -1 & 1 & -1 & -1 \end{bmatrix}}_{\overline{c}^\mathsf{T}} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\overline{v}_1} = 0 < \underbrace{\begin{bmatrix} 1 & -1 & 1 & -1 & -1 \end{bmatrix}}_{\overline{c}^\mathsf{T}} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\overline{v}_2} = 1 \, .$$

Now $R = [7; 2; 3; 4; 5]$ (rows of A in A_2).

Example (cont):

Solve

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 \end{bmatrix}}_{A_2{}^{\mathsf{T}}} \overline{u}_2 = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \end{bmatrix}}_{\overline{c}}$$

for $\overline{u}_2$ yielding $\overline{u}_2 = [1\ 0\ 0\ 0\ 0]^{\mathsf{T}}$. Since $\overline{u}_2 \geq 0$, we are in Case 1: we have found an optimum point, $\overline{v}_2$, with optimal value 1.

Since we have that $v_F = \overline{1}^{\mathsf{T}} \overline{e}_2 = 1$, the equality of the optimial point and v_F implies that

$$F :\ x + y \geq 1\ \wedge\ x - y \geq -1$$

is $T_{\mathbb{Q}}$-satisfiable. In particular, extract from

$$\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z \end{bmatrix} = \overline{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

the assignment

$$x = x_1 - x_2 = 1 - 0 = 1 \quad \text{and} \quad y = y_1 - y_2 = 0 - 0 = 0 \ ,$$

Example: $\Sigma_{\mathbb{Q}}$-formula (7.1)

$$F:\ x \geq 0\ \wedge\ y \geq 0\ \wedge\ x \geq 2\ \wedge\ y \geq 2\ \wedge\ x + y \leq 3\ ,$$

or, in matrix form,

$$F:\ \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ -2 \\ -2 \\ 3 \end{bmatrix}$$

Is F $T_{\mathbb{Q}}$-satisfiable?

Step 0

Because x and y are already constrained to be nonnegative, we do not need to introduce new x_1, x_2, y_1, y_2. Rewrite:

$$\underbrace{[1\ \ 1]}_{D_1} \begin{bmatrix} x \\ y \end{bmatrix} \leq \underbrace{[3]}_{\overline{e}_1} \quad \text{and} \quad \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{D_2} \begin{bmatrix} x \\ y \end{bmatrix} \geq \underbrace{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}_{\overline{e}_2}$$

so that $\overline{e}_1 \geq 0$ and $\overline{e}_2 > 0$.

Example (cont)

Then (7.2):

$$\begin{array}{rcl} \textbf{max} & \overline{1}^{\mathsf{T}}(D_2\overline{x}-\overline{z}) & \\ \textbf{subject to} & & \\ \overline{x},\overline{z} & \geq & \overline{0} \\ D_1\overline{x} & \leq & \overline{e}_1 \\ D_2\overline{x}-\overline{z} & \leq & \overline{e}_2 \end{array}$$

Expanding, we have

$$\overline{c}^{\mathsf{T}}\overline{x} = \overline{1}^{\mathsf{T}}\left[\begin{array}{cc} D_2 & -I \end{array}\right]\left[\begin{array}{c} x \\ y \\ z_1 \\ z_2 \end{array}\right] = [1\ 1]\left[\begin{array}{cccr} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right]\left[\begin{array}{c} x \\ y \\ z_1 \\ z_2 \end{array}\right]$$

$$= \underbrace{\left[\begin{array}{cccc} 1 & 1 & -1 & -1 \end{array}\right]}_{\overline{c}^{\mathsf{T}}}\left[\begin{array}{c} x \\ y \\ z_1 \\ z_2 \end{array}\right]$$

Example (cont):

obtaining the optimization problem (7.3)

$$\textbf{max} \quad \underbrace{[1\ 1\ -1\ -1]}_{\overline{c}^{\mathsf{T}}} \begin{bmatrix} x \\ y \\ z_1 \\ z_2 \end{bmatrix}$$

subject to

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z_1 \\ z_2 \end{bmatrix} \leq \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 2 \\ 2 \end{bmatrix}}_{\overline{b}}$$

Example (cont):

Use the initial vertex

$$\overline{v}_1 = \begin{bmatrix} x \\ y \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

in **Step 1**.

F is satisfiable iff the optimal value v_F is equal to

$$\overline{1}^{\mathsf{T}}\overline{e}_2 = [1\ 1] \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 4\ .$$

We use the simplex algorithm to find the optimum.

Example (cont):

Step 1

Choose rows $R = [1; 2; 3; 4]$ of A and $\overline{b}$, giving

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\overline{v}_1} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\overline{b}_1}$$

Solving

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{{A_1}^{\mathsf{T}}} \overline{u}_1 = \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}}_{\overline{c}}$$

yields $\overline{u}_1 = [-1\ -1\ 1\ 1]^{\mathsf{T}}$. Adding 0s for the rows not in R produces $\overline{u}$:

$$\overline{u} = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}^{\mathsf{T}}.$$

Example (cont):

Since $u_1, u_2 < 0$, we are in Case 2 with $k = k' = 1$. Let $\overline{y}$ be the first column of $-A_1^{-1}$: solve

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{A_1} \overline{y} = \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{-\overline{e}_1}$$

to yield $\overline{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^{\mathsf{T}}$. Then $S = [5; 6]$; *i.e.*, the 5th and 6th rows $\overline{a}$ of A are such that $\overline{ay} > 0$. Choose the largest λ_1 such that

$$A(\overline{v}_1 + \lambda_1 \overline{y}) \leq \overline{b}$$

Example (cont):

Focusing on the 5th and 6th rows of A (since $S' = [5;6]$), choose the largest λ_1 such that

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}}_{\text{rows 5,6 of } A} \left(\underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\overline{v}_1} + \lambda_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\overline{y}} \right) \leq \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{\text{rows 5,6 of } \overline{b}}$$

Namely, choose $\lambda_1 = 2$ (and $\ell = 6$). Then

$$\overline{v}_2 = \overline{v}_1 + \lambda_1 \overline{y} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example (cont):

Replace the 1st row of A_1 (since $k' = 1$) by the 6th row of A (since $\ell = 6$) to produce

$$A_2 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \overline{b}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Have we made progress? Yes, for

$$\overline{c}^{\mathsf{T}} \overline{v}_1 = 0 < 2 = \overline{c}^{\mathsf{T}} \overline{v}_2 \ .$$

The objective function has increased from 0 to 2.

Example (cont):

Step 2

Now $R = [6; 2; 3; 4]$ (the indices of rows of A in A_2). Solve

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{{A_2}^{\mathsf{T}}} \overline{u}_2 = \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}}_{\overline{c}}$$

to yield $\overline{u}_2 = \begin{array}{cccccc} [& 1 & -1 & 0 & 1 &]^{\mathsf{T}} \\ & 6 & 2 & 3 & 4 & \end{array}$. Then filling in 0s for the other rows of A produces:

$$\overline{u} = \begin{array}{ccccccc} [0 & -1 & 0 & 1 & 0 & 1 & 0]^{\mathsf{T}} \\ & 2 & 3 & 4 & & 6 & \end{array}$$

$u_2 < 0$, so $k = 2$, which corresponds to row $k' = 2$ of $\overline{u}_2$. According to Case 2, let $\overline{y}$ be the 2nd column of $-A_2^{-1}$: solve $A_2\overline{y} = -\mathrm{e}_2$ to yield $\overline{y} = [0\ 1\ 0\ 0]^{\mathsf{T}}$. Then the 5th and 7th rows $\overline{a}$ of A are such that $\overline{ay} > 0$ so that $S = [5; 7]$.

Example (cont):

Focusing on the 5th and 7th rows of A, choose the largest λ_2 such that

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}}_{\text{rows 5,7 of } A} \left(\underbrace{\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\overline{v}_2} + \lambda_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\overline{b}} \right) \leq \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{\text{rows 5,7 of } \overline{b}}$$

Choose $\lambda_2 = 1$ (and $\ell = 5$). Then

$$\overline{v}_3 = \overline{v}_2 + \lambda_2 \overline{y} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Example (cont):

Replace the 2nd row of A_2 (since $k' = 2$) by the 5th row of A (since $\ell = 5$) to produce

$$A_3 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \overline{b}_3 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

Have we made progress? Yes, for

$$\overline{c}^{\mathsf{T}}\overline{v}_1 = 0 < \overline{c}^{\mathsf{T}}\overline{v}_2 = 2 < \overline{c}^{\mathsf{T}}\overline{v}_3 = 3 \ .$$

The objective function has increased from 2 to 3.

Example (cont):

Step 3

Now $R = [6;5;3;4]$. Solve ${A_3}^{\mathsf{T}}\overline{u}_3 = \overline{c}$, yielding $\overline{u}_3 = [0\ 1\ 1\ 1]^{\mathsf{T}}$. Now $\overline{u}_3 \geq \overline{0}$, so we are in Case 1: $\overline{v}_3$ is the optimum with objective value

$$\underbrace{[1\ 1\ \ -1\ \ -1]}_{\overline{c}^{\mathsf{T}}} \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\overline{v}_3} = 3\ .$$

Final Step: Satisfiability

The optimal value of the constructed optimization problem is 3, which is less than the required $v_F = 4$ of **Step 0**. Hence, F is $T_{\mathbb{Q}}$-unsatisfiable.