

The Theory of Equality T_E

$$\Sigma_E : \{=, a, b, c, \dots, f, g, h, \dots, p, q, r, \dots\}$$

uninterpreted symbols:

- constants a, b, c, \dots
- functions f, g, h, \dots
- predicates p, q, r, \dots

Example:

$$x = y \wedge f(x) \neq f(y) \quad T_E\text{-unsatisfiable}$$

$$f(x) = f(y) \wedge x \neq y \quad T_E\text{-unsatisfiable}$$

$$f(f(f(a))) = a \wedge f(f(f(f(f(a)))))) = a \wedge f(a) \neq a \quad T_E\text{-unsatisfiable}$$

9. Quantifier-free Equality and Data Structures

Axioms of T_E

1. $\forall x. x = x$ (reflexivity)
2. $\forall x, y. x = y \rightarrow y = x$ (symmetry)
3. $\forall x, y, z. x = y \wedge y = z \rightarrow x = z$ (transitivity)

define $=$ to be an equivalence relation.

Axiom schema

4. for each positive integer n and n -ary function symbol f ,

$$\forall x_1, \dots, x_n, y_1, \dots, y_n. \bigwedge_i x_i = y_i \rightarrow f(x_1, \dots, x_n) = f(y_1, \dots, y_n) \quad (\text{congruence})$$

For example,

$$\forall x, y. x = y \rightarrow f(x) = f(y)$$

Then

$$x = g(y, z) \rightarrow f(x) = f(g(y, z))$$

is T_E -valid.

Axiom schema

5. for each positive integer n and n -ary predicate symbol p ,

$$\forall x_1, \dots, x_n, y_1, \dots, y_n. \bigwedge_i x_i = y_i \rightarrow (p(x_1, \dots, x_n) \leftrightarrow p(y_1, \dots, y_n)) \quad (\text{equivalence})$$

Thus,

$$x = y \rightarrow (p(x) \leftrightarrow p(y))$$

is T_E -valid.

We discuss T_E -formulae without predicates

For example, for Σ_E -formula

$$F : p(x) \wedge q(x, y) \wedge q(y, z) \rightarrow \neg q(x, z)$$

introduce fresh constant \bullet and fresh functions f_p and f_q , and transform F to

$$G : f_p(x) = \bullet \wedge f_q(x, y) = \bullet \wedge f_q(y, z) = \bullet \rightarrow f_q(x, z) \neq \bullet .$$

Equivalence and Congruence Relations: Basics

Binary relation R over set S

- is an equivalence relation if
 - ▶ reflexive: $\forall s \in S. sRs$;
 - ▶ symmetric: $\forall s_1, s_2 \in S. s_1Rs_2 \rightarrow s_2Rs_1$;
 - ▶ transitive: $\forall s_1, s_2, s_3 \in S. s_1Rs_2 \wedge s_2Rs_3 \rightarrow s_1Rs_3$.

Example:

Define the binary relation \equiv_2 over the set \mathbb{Z} of integers

$$m \equiv_2 n \text{ iff } (m \bmod 2) = (n \bmod 2)$$

That is, $m, n \in \mathbb{Z}$ are related iff they are both even or both odd.

\equiv_2 is an equivalence relation

- is a congruence relation if in addition

$$\forall \vec{s}, \vec{t}. \bigwedge_{i=1}^n s_i R t_i \rightarrow f(\vec{s}) R f(\vec{t}) .$$

Classes

For $\left\{ \begin{array}{l} \text{equivalence} \\ \text{congruence} \end{array} \right\}$ relation R over set S ,

The $\left\{ \begin{array}{l} \text{equivalence} \\ \text{congruence} \end{array} \right\}$ class of $s \in S$ under R is

$$[s]_R \stackrel{\text{def}}{=} \{s' \in S : sRs'\} .$$

Example:

The equivalence class of 3 under \equiv_2 over \mathbb{Z} is

$$[3]_{\equiv_2} = \{n \in \mathbb{Z} : n \text{ is odd}\} .$$

Partitions

A partition P of S is a set of subsets of S that is

- ▶ total $\left(\bigcup_{S' \in P} S' \right) = S$
- ▶ disjoint $\forall S_1, S_2 \in P. S_1 \cap S_2 = \emptyset$

Quotient

The quotient S/R of S by $\left\{ \begin{array}{l} \text{equivalence} \\ \text{congruence} \end{array} \right\}$ relation R is the set of $\left\{ \begin{array}{l} \text{equivalence} \\ \text{congruence} \end{array} \right\}$ classes

$$S/R = \{[s]_R : s \in S\} .$$

It is a partition

Example: The quotient \mathbb{Z}/\equiv_2 is a partition of \mathbb{Z} . The set of equivalence classes

$$\{\{n \in \mathbb{Z} : n \text{ is odd}\}, \{n \in \mathbb{Z} : n \text{ is even}\}\}$$

Note duality between relations and classes

Refinements

Two binary relations R_1 and R_2 over set S .

R_1 is refinement of R_2 , $R_1 \prec R_2$, if

$$\forall s_1, s_2 \in S. s_1 R_1 s_2 \rightarrow s_1 R_2 s_2 .$$

R_1 refines R_2 .

Examples:

▶ For $S = \{a, b\}$,

$$R_1 : \{aR_1b\} \quad R_2 : \{aR_2b, bR_2b\}$$

Then $R_1 \prec R_2$

▶ For set S ,

$$R_1 \text{ induced by the partition } P_1 : \{\{s\} : s \in S\}$$

$$R_2 \text{ induced by the partition } P_2 : \{S\}$$

Then $R_1 \prec R_2$.

▶ For set \mathbb{Z}

$$R_1 : \{xR_1y : x \bmod 2 = y \bmod 2\}$$

$$R_2 : \{xR_2y : x \bmod 4 = y \bmod 4\}$$

Then $R_2 \prec R_1$.



Closures

Given binary relation R over S .

The equivalence closure R^E of R is the equivalence relation s.t.

- ▶ R refines R^E , i.e. $R \prec R^E$;
- ▶ for all other equivalence relations R' s.t. $R \prec R'$, either $R' = R^E$ or $R^E \prec R'$

That is, R^E is the “smallest” equivalence relation that “covers” R .

Example: If $S = \{a, b, c, d\}$ and $R = \{aRb, bRc, dRd\}$, then

- $aRb, bRc, dRd \in R^E$ since $R \subseteq R^E$;
- $aRa, bRb, cRc \in R^E$ by reflexivity;
- $bRa, cRb \in R^E$ by symmetry;
- $aRc \in R^E$ by transitivity;
- $cRa \in R^E$ by symmetry.

Hence,

$$R^E = \{aRb, bRa, aRa, bRb, bRc, cRb, cRc, aRc, cRa, dRd\} .$$

Similarly, the congruence closure R^C of R is the “smallest” congruence relation that “covers” R .



Congruence Closure Algorithm

Given Σ_E -formula

$$F : s_1 = t_1 \wedge \dots \wedge s_m = t_m \wedge s_{m+1} \neq t_{m+1} \wedge \dots \wedge s_n \neq t_n$$

decide if F is Σ_E -satisfiable.

Definition: For Σ_E -formula F ,

the subterm set S_F of F is the set that contains precisely the subterms of F .

Example: The subterm set of

$$F : f(a, b) = a \wedge f(f(a, b), b) \neq a$$

is

$$S_F = \{a, b, f(a, b), f(f(a, b), b)\} .$$



The Algorithm

Given Σ_E -formula F

$$F : s_1 = t_1 \wedge \dots \wedge s_m = t_m \wedge s_{m+1} \neq t_{m+1} \wedge \dots \wedge s_n \neq t_n$$

with subterm set S_F , F is T_E -satisfiable iff there exists a congruence relation \sim over S_F such that

- ▶ for each $i \in \{1, \dots, m\}$, $s_i \sim t_i$;
- ▶ for each $i \in \{m+1, \dots, n\}$, $s_i \not\sim t_i$.

Such congruence relation \sim defines T_E -interpretation $I : (D_I, \alpha_I)$ of F . D_I consists of $|S_F / \sim|$ elements, one for each congruence class of S_F under \sim .

Instead of writing $I \models F$ for this T_E -interpretation, we abbreviate $\sim \models F$

The goal of the algorithm is to construct the congruence relation of S_F , or to prove that no congruence relation exists.



$$F : \underbrace{s_1 = t_1 \wedge \dots \wedge s_m = t_m}_{\text{generate congruence closure}} \wedge \underbrace{s_{m+1} \neq t_{m+1} \wedge \dots \wedge s_n \neq t_n}_{\text{search for contradiction}}$$

The algorithm performs the following steps:

1. Construct the congruence closure \sim of

$$\{s_1 = t_1, \dots, s_m = t_m\}$$

over the subterm set S_F . Then

$$\sim \models s_1 = t_1 \wedge \dots \wedge s_m = t_m .$$

2. If for any $i \in \{m+1, \dots, n\}$, $s_i \sim t_i$, return unsatisfiable.
3. Otherwise, $\sim \models F$, so return satisfiable.

How do we actually construct the congruence closure in Step 1?

Initially, begin with the finest congruence relation \sim_0 given by the partition

$$\{\{s\} : s \in S_F\} .$$

That is, let each term of S_F be its own congruence class.

Then, for each $i \in \{1, \dots, m\}$, impose $s_i = t_i$ by merging the congruence classes

$$[s_i]_{\sim_{i-1}} \quad \text{and} \quad [t_i]_{\sim_{i-1}}$$

to form a new congruence relation \sim_i . To accomplish this merging,

- ▶ form the union of $[s_i]_{\sim_{i-1}}$ and $[t_i]_{\sim_{i-1}}$
- ▶ propagate any new congruences that arise within this union.

The new relation \sim_i is a congruence relation in which $s_i \sim t_i$.

Example: Given Σ_E -formula

$$F : f(a, b) = a \wedge f(f(a, b), b) \neq a$$

Construct initial partition by letting each member of the subterm set S_F be its own class:

1. $\{\{a\}, \{b\}, \{f(a, b)\}, \{f(f(a, b), b)\}\}$

According to the first literal $f(a, b) = a$, merge

$$\{f(a, b)\} \quad \text{and} \quad \{a\}$$

to form partition

2. $\{\{a, f(a, b)\}, \{b\}, \{f(f(a, b), b)\}\}$

According to the (congruence) axiom,

$$f(a, b) \sim a, \quad b \sim b \quad \text{implies} \quad f(f(a, b), b) \sim f(a, b) ,$$

resulting in the new partition

3. $\{\{a, f(a, b), f(f(a, b), b)\}, \{b\}\}$

This partition represents the congruence closure of S_F . Now, is it the case that

4. $\{\{a, f(a, b), f(f(a, b), b)\}, \{b\}\} \models F$?

No, as $f(f(a, b), b) \sim a$ but F asserts that $f(f(a, b), b) \neq a$.

Hence, F is T_E -unsatisfiable.

Example: Given Σ_E -formula

$$F : f(f(f(a))) = a \wedge f(f(f(f(f(a)))))) = a \wedge f(a) \neq a$$

From the subterm set S_F , the initial partition is

1. $\{\{a\}, \{f(a)\}, \{f^2(a)\}, \{f^3(a)\}, \{f^4(a)\}, \{f^5(a)\}\}$

where, for example, $f^3(a)$ abbreviates $f(f(f(a)))$.

According to the literal $f^3(a) = a$, merge

$$\{f^3(a)\} \quad \text{and} \quad \{a\} .$$

From the union,

2. $\{\{a, f^3(a)\}, \{f(a)\}, \{f^2(a)\}, \{f^4(a)\}, \{f^5(a)\}\}$

deduce the following congruence propagations:

$$f^3(a) \sim a \Rightarrow f(f^3(a)) \sim f(a) \quad \text{i.e.} \quad f^4(a) \sim f(a)$$

and

$$f^4(a) \sim f(a) \Rightarrow f(f^4(a)) \sim f(f(a)) \quad \text{i.e.} \quad f^5(a) \sim f^2(a)$$

Thus, the final partition for this iteration is the following:

3. $\{\{a, f^3(a)\}, \{f(a), f^4(a)\}, \{f^2(a), f^5(a)\}\} .$

$$3. \{ \{a, f^3(a)\}, \{f(a), f^4(a)\}, \{f^2(a), f^5(a)\} \} .$$

From the second literal, $f^5(a) = a$, merge

$$\{f^2(a), f^5(a)\} \text{ and } \{a, f^3(a)\}$$

to form the partition

$$4. \{ \{a, f^2(a), f^3(a), f^5(a)\}, \{f(a), f^4(a)\} \} .$$

Propagating the congruence

$$f^3(a) \sim f^2(a) \Rightarrow f(f^3(a)) \sim f(f^2(a)) \text{ i.e. } f^4(a) \sim f^3(a)$$

yields the partition

$$5. \{ \{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a)\} \} ,$$

which represents the congruence closure in which all of S_F are equal. Now,

$$6. \{ \{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a)\} \} \models F ?$$

No, as $f(a) \sim a$, but F asserts that $f(a) \neq a$. Hence, F is T_E -unsatisfiable.

Example: Given Σ_E -formula

$$F : f(x) = f(y) \wedge x \neq y .$$

The subterm set S_F induces the following initial partition:

$$1. \{ \{x\}, \{y\}, \{f(x)\}, \{f(y)\} \} .$$

Then $f(x) = f(y)$ indicates to merge

$$\{f(x)\} \text{ and } \{f(y)\} .$$

The union $\{f(x), f(y)\}$ does not yield any new congruences, so the final partition is

$$2. \{ \{x\}, \{y\}, \{f(x), f(y)\} \} .$$

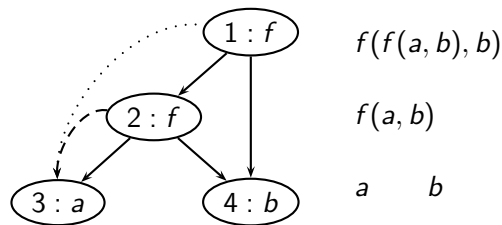
Does

$$3. \{ \{x\}, \{y\}, \{f(x), f(y)\} \} \models F ?$$

Yes, as $x \not\sim y$, agreeing with $x \neq y$. Hence, F is T_E -satisfiable.

Directed Acyclic Graph (DAG)

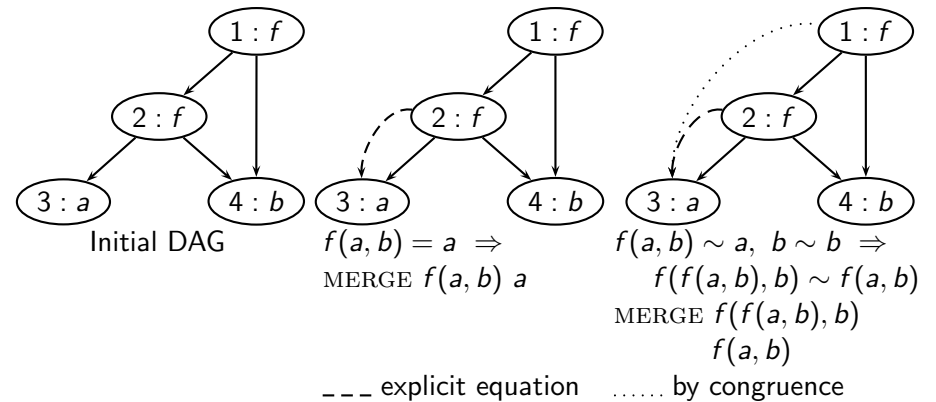
For Σ_E -formula F , graph-based data structure for representing the subterms of S_F (and congruence relation between them).



Efficient way for computing the congruence closure algorithm.

T_E -Satisfiability (Summary of idea)

$$f(a, b) = a \wedge f(f(a, b), b) \neq a$$



$$\left. \begin{array}{l} \text{FIND } f(f(a, b), b) = a = \text{FIND } a \\ f(f(a, b), b) \neq a \end{array} \right\} \Rightarrow \text{Unsatisfiable}$$

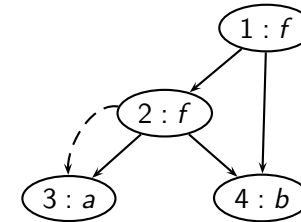
DAG representation

```
type node = {  
  id      : id      node's unique identification number  
  fn      : string  constant or function name  
  args    : id list list of function arguments  
  mutable find : id  the representative of the congruence class  
  mutable ccpair : id set if the node is the representative for its congruence class, then its ccpair (congruence closure parents) are all parents of nodes in its congruence class  
}
```

9-21

DAG Representation of node 2

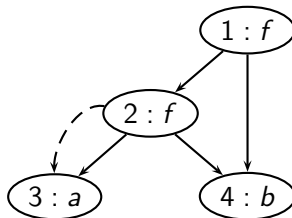
```
type node = {  
  id      : id      ... 2  
  fn      : string  ... f  
  args    : id list ... [3,4]  
  mutable find : id  ... 3  
  mutable ccpair : id set ... ∅  
}
```



9-22

DAG Representation of node 3

```
type node = {  
  id      : id      ... 3  
  fn      : string  ... a  
  args    : id list ... []  
  mutable find : id  ... 3  
  mutable ccpair : id set ... {1,2}  
}
```



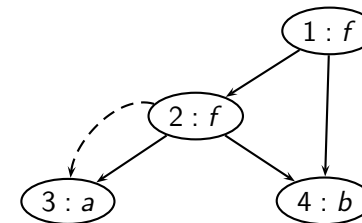
9-23

The Implementation

FIND function

returns the representative of node's congruence class

```
let rec FIND i =  
  let n = NODE i in  
  if n.find = i then i else FIND n.find
```



Example: FIND 2 = 3
 FIND 3 = 3
3 is the representative of 2.

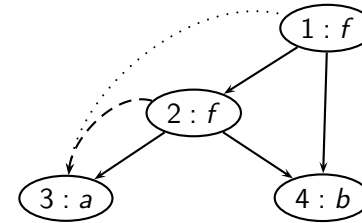
9-24

UNION function

```
let UNION  $i_1$   $i_2$  =  
  let  $n_1$  = NODE (FIND  $i_1$ ) in  
  let  $n_2$  = NODE (FIND  $i_2$ ) in  
   $n_1$ .find  $\leftarrow$   $n_2$ .find;  
   $n_2$ .ccpar  $\leftarrow$   $n_1$ .ccpar  $\cup$   $n_2$ .ccpar;  
   $n_1$ .ccpar  $\leftarrow$   $\emptyset$ 
```

n_2 is the representative of the union class

Example



```
UNION 1 2       $n_1 = 1$     $n_2 = 3$   
  1.find  $\leftarrow$  3  
  3.ccpar  $\leftarrow$  {1, 2}  
  1.ccpar  $\leftarrow$   $\emptyset$ 
```

CCPAR function

Returns parents of all nodes in i 's congruence class

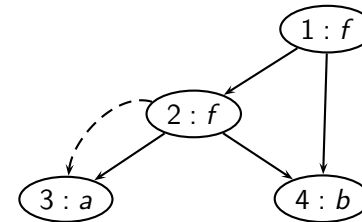
```
let CCPAR  $i$  =  
  (NODE (FIND  $i$ )).ccpar
```

CONGRUENT predicate

Test whether i_1 and i_2 are congruent

```
let CONGRUENT  $i_1$   $i_2$  =  
  let  $n_1$  = NODE  $i_1$  in  
  let  $n_2$  = NODE  $i_2$  in  
   $n_1$ .fn =  $n_2$ .fn  
   $\wedge$   $|n_1$ .args| =  $|n_2$ .args|  
   $\wedge \forall i \in \{1, \dots, |n_1$ .args|\}. FIND  $n_1$ .args[ $i$ ] = FIND  $n_2$ .args[ $i$ ]
```

Example:



Are 1 and 2 congruent?

fn fields	— both f
# of arguments	— same
left arguments $f(a, b)$ and a	— both congruent to 3
right arguments b and b	— both 4 (congruent)

Therefore 1 and 2 are congruent.

MERGE function

```

let rec MERGE  $i_1 i_2 =$ 
  if FIND  $i_1 \neq$  FIND  $i_2$  then begin
    let  $P_{i_1} =$  CCPAR  $i_1$  in
    let  $P_{i_2} =$  CCPAR  $i_2$  in
    UNION  $i_1 i_2$ ;
    foreach  $t_1, t_2 \in P_{i_1} \times P_{i_2}$  do
      if FIND  $t_1 \neq$  FIND  $t_2 \wedge$  CONGRUENT  $t_1 t_2$ 
      then MERGE  $t_1 t_2$ 
    done
  end

```

P_{i_1} and P_{i_2} store the current values of CCPAR i_1 and CCPAR i_2 .

Decision Procedure: T_E -satisfiability

Given Σ_E -formula

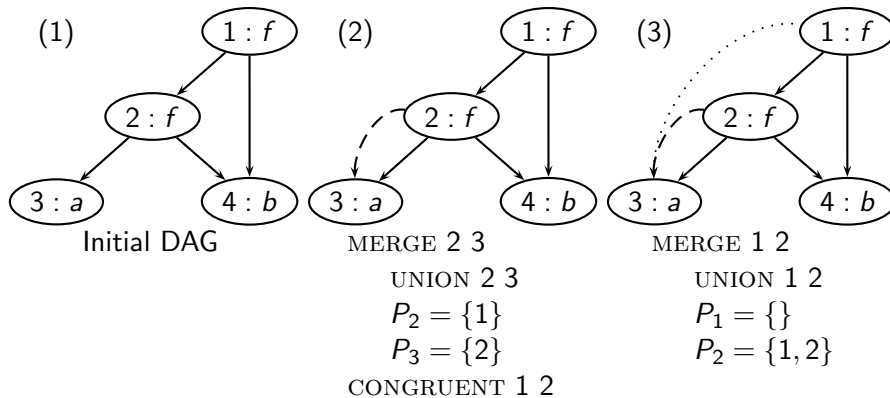
$$F : s_1 = t_1 \wedge \dots \wedge s_m = t_m \wedge s_{m+1} \neq t_{m+1} \wedge \dots \wedge s_n \neq t_n,$$

with subterm set S_F , perform the following steps:

1. Construct the initial DAG for the subterm set S_F .
2. For $i \in \{1, \dots, m\}$, MERGE $s_i t_i$.
3. If FIND $s_i =$ FIND t_i for some $i \in \{m+1, \dots, n\}$, return unsatisfiable.
4. Otherwise (if FIND $s_i \neq$ FIND t_i for all $i \in \{m+1, \dots, n\}$) return satisfiable.

Example 1: T_E -Satisfiability

$$f(a, b) = a \wedge f(f(a, b), b) \neq a$$



FIND $f(f(a, b), b) = a =$ FIND $a \Rightarrow$ **Unsatisfiable**

Given Σ_E -formula

$$F : f(a, b) = a \wedge f(f(a, b), b) \neq a.$$

The subterm set is

$$S_F = \{a, b, f(a, b), f(f(a, b), b)\},$$

resulting in the initial partition

$$(1) \{\{a\}, \{b\}, \{f(a, b)\}, \{f(f(a, b), b)\}\}$$

in which each term is its own congruence class. Fig (1).

Final partition

$$(2) \{\{a, f(a, b), f(f(a, b), b)\}, \{b\}\}$$

Note: dash edge ----- merge dictated by equalities in F
dotted edge deduced merge

Does

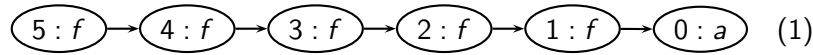
$$(3) \{\{a, f(a, b), f(f(a, b), b)\}, \{b\}\} \models F ?$$

No, as $f(f(a, b), b) \sim a$, but F asserts that $f(f(a, b), b) \neq a$.

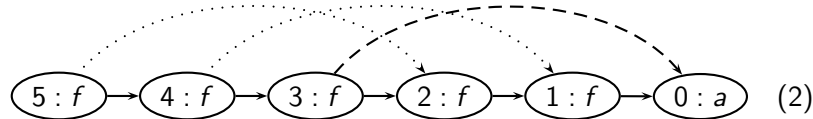
Hence, F is T_E -unsatisfiable.

Example 2: T_E -Satisfiability

$$f(f(f(a))) = a \wedge f(f(f(f(f(a)))))) = a \wedge f(a) \neq a$$



Initial DAG



$$\begin{aligned} f(f(f(a))) = a &\Rightarrow \text{MERGE } 3 \ 0 & P_3 = \{4\} & P_0 = \{1\} \\ &\Rightarrow \text{MERGE } 4 \ 1 & P_4 = \{5\} & P_1 = \{2\} \\ &\Rightarrow \text{MERGE } 5 \ 2 & P_5 = \{\} & P_2 = \{3\} \end{aligned}$$

Given Σ_E -formula

$$F: f(f(f(a))) = a \wedge f(f(f(f(f(a)))))) = a \wedge f(a) \neq a,$$

which induces the initial partition

1. $\{\{a\}, \{f(a)\}, \{f^2(a)\}, \{f^3(a)\}, \{f^4(a)\}, \{f^5(a)\}\}$.

The equality $f^3(a) = a$ induces the partition

2. $\{\{a, f^3(a)\}, \{f(a), f^4(a)\}, \{f^2(a), f^5(a)\}\}$.

The equality $f^5(a) = a$ induces the partition

3. $\{\{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a)\}\}$.

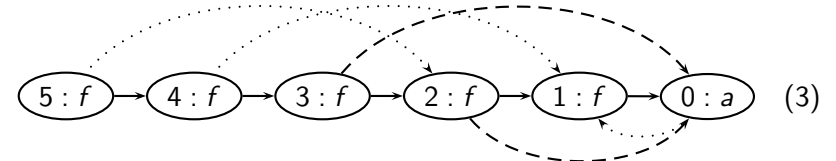
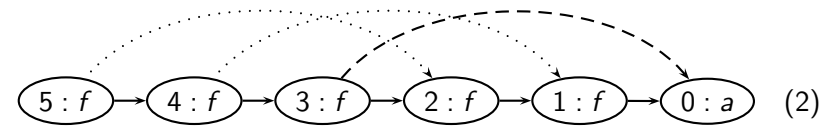
Now, does

$$\{\{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a)\}\} \models F?$$

No, as $f(a) \sim a$, but F asserts that $f(a) \neq a$. Hence, F is T_E -unsatisfiable.

Example 2: T_E -Satisfiability

$$f(f(f(a))) = a \wedge f(f(f(f(f(a)))))) = a \wedge f(a) \neq a$$



$$\begin{aligned} f(f(f(f(f(a)))))) = a &\Rightarrow \text{MERGE } 5 \ 0 & P_5 = \{3\} & P_0 = \{1, 4\} \\ &\Rightarrow \text{MERGE } 3 \ 1 & \text{STOP. Why?} \end{aligned}$$

FIND $f(a) = f(a) = \text{FIND } a \Rightarrow$ **Unsatisfiable**

Theorem (Sound and Complete)

Quantifier-free conjunctive Σ_E -formula F is T_E -satisfiable iff the congruence closure algorithm returns satisfiable.

Recursive Data Structures

Quantifier-free Theory of Lists T_{cons}

$$\Sigma_{\text{cons}} : \{\text{cons}, \text{car}, \text{cdr}, \text{atom}, =\}$$

- constructor cons : $\text{cons}(a, b)$ list constructed by prepending a to b
- left projector car : $\text{car}(\text{cons}(a, b)) = a$
- right projector cdr : $\text{cdr}(\text{cons}(a, b)) = b$
- atom : unary predicate

Axioms of T_{cons}

- ▶ reflexivity, symmetry, transitivity
- ▶ congruence axioms:

$$\begin{aligned} \forall x_1, x_2, y_1, y_2. x_1 = x_2 \wedge y_1 = y_2 &\rightarrow \text{cons}(x_1, y_1) = \text{cons}(x_2, y_2) \\ \forall x, y. x = y &\rightarrow \text{car}(x) = \text{car}(y) \\ \forall x, y. x = y &\rightarrow \text{cdr}(x) = \text{cdr}(y) \end{aligned}$$

- ▶ equivalence axiom:

$$\forall x, y. x = y \rightarrow (\text{atom}(x) \leftrightarrow \text{atom}(y))$$

- ▶
 - (A1) $\forall x, y. \text{car}(\text{cons}(x, y)) = x$ (left projection)
 - (A2) $\forall x, y. \text{cdr}(\text{cons}(x, y)) = y$ (right projection)
 - (A3) $\forall x. \neg \text{atom}(x) \rightarrow \text{cons}(\text{car}(x), \text{cdr}(x)) = x$ (construction)
 - (A4) $\forall x, y. \neg \text{atom}(\text{cons}(x, y))$ (atom)

Simplifications

- ▶ Consider only quantifier-free conjunctive Σ_{cons} -formulae. Convert non-conjunctive formula to DNF and check each disjunct.
- ▶ $\neg \text{atom}(u_i)$ literals are removed:

replace $\neg \text{atom}(u_i)$ with $u_i = \text{cons}(u_i^1, u_i^2)$

 by the (construction) axiom.
- ▶ Because of similarity to $\Sigma_{=}$, we sometimes combine $\Sigma_{\text{cons}} \cup \Sigma_{=}$.

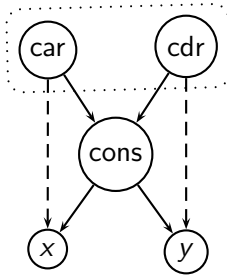
Algorithm: T_{cons} -Satisfiability (the idea)

$$\begin{aligned} F : & \underbrace{s_1 = t_1 \wedge \dots \wedge s_m = t_m}_{\text{generate congruence closure}} \\ & \wedge \underbrace{s_{m+1} \neq t_{m+1} \wedge \dots \wedge s_n \neq t_n}_{\text{search for contradiction}} \\ & \wedge \underbrace{\text{atom}(u_1) \wedge \dots \wedge \text{atom}(u_\ell)}_{\text{search for contradiction}} \end{aligned}$$

where $s_i, t_i,$ and u_i are T_{cons} -terms

Algorithm: T_{cons} -Satisfiability

1. Construct the initial DAG for S_F
2. for each node n with $n.\text{fn} = \text{cons}$
 - ▶ add $\text{car}(n)$ and MERGE $\text{car}(n) n.\text{args}[1]$
 - ▶ add $\text{cdr}(n)$ and MERGE $\text{cdr}(n) n.\text{args}[2]$
 by axioms (A1), (A2)
3. for $1 \leq i \leq m$, MERGE $s_i t_i$
4. for $m + 1 \leq i \leq n$, if FIND $s_i = \text{FIND } t_i$, return **unsatisfiable**
5. for $1 \leq i \leq \ell$, if $\exists v. \text{FIND } v = \text{FIND } u_i \wedge v.\text{fn} = \text{cons}$, return **unsatisfiable**
6. Otherwise, return **satisfiable**



Example:

Given $(\Sigma_{\text{cons}} \cup \Sigma_{=})$ -formula

$$F : \quad \text{car}(x) = \text{car}(y) \wedge \text{cdr}(x) = \text{cdr}(y) \\ \wedge \neg \text{atom}(x) \wedge \neg \text{atom}(y) \wedge f(x) \neq f(y)$$

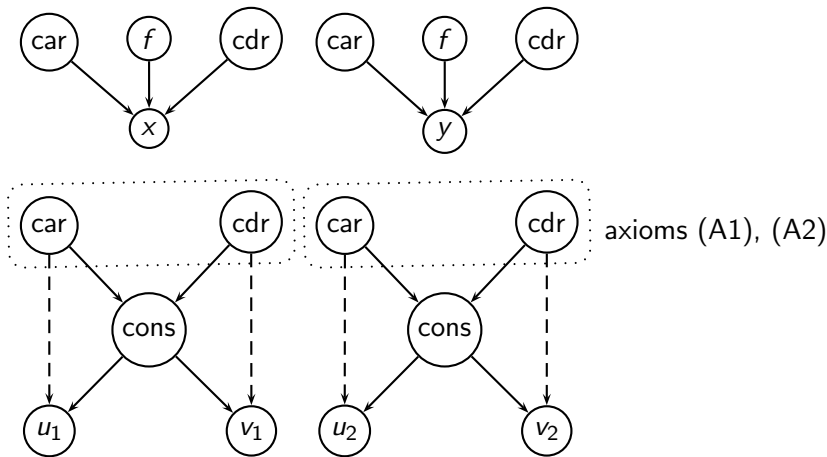
where the function symbol f is in $\Sigma_{=}$

- (1) $\text{car}(x) = \text{car}(y) \quad \wedge$
- (2) $\text{cdr}(x) = \text{cdr}(y) \quad \wedge$
- (3) $x = \text{cons}(u_1, v_1) \quad \wedge$
- (4) $y = \text{cons}(u_2, v_2) \quad \wedge$
- (5) $f(x) \neq f(y)$

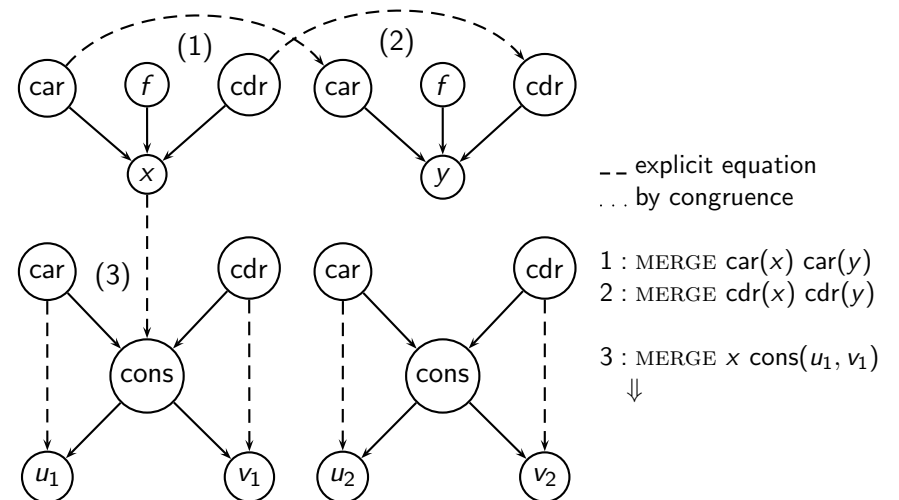
Recall the projection axioms:

- (A1) $\forall x, y. \text{car}(\text{cons}(x, y)) = x$
- (A2) $\forall x, y. \text{cdr}(\text{cons}(x, y)) = y$

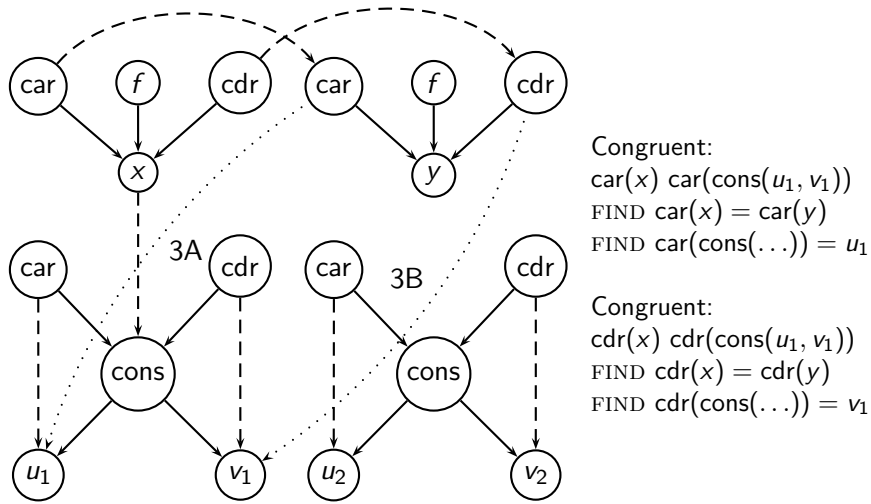
Example(cont): Initial DAG



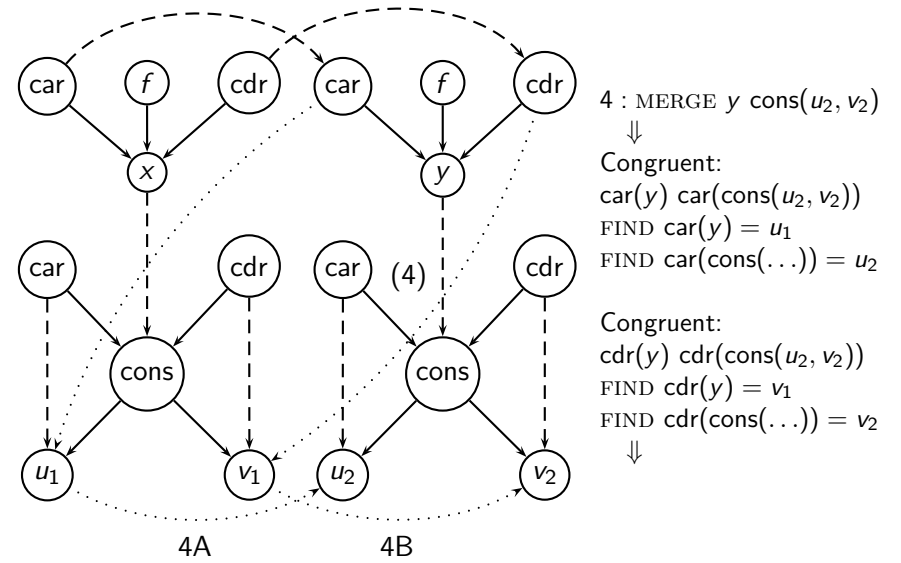
Example(cont): MERGE



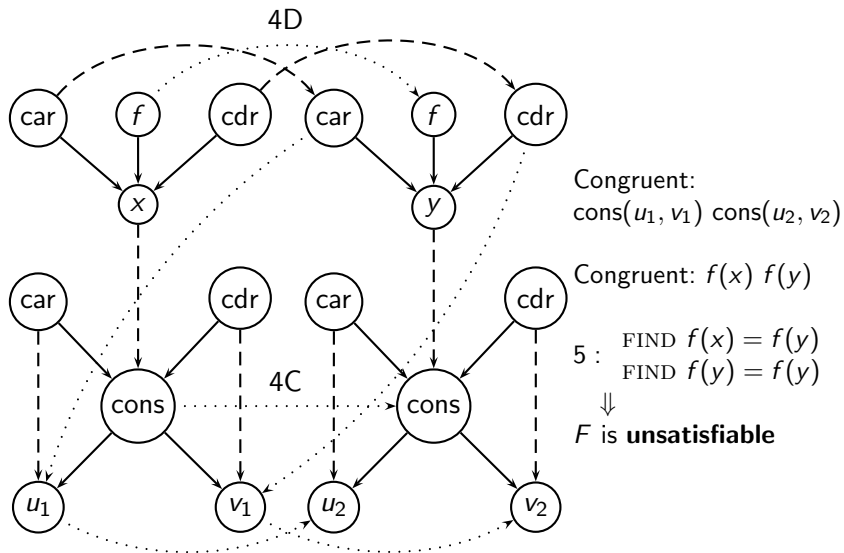
Example(cont): Propagation



Example(cont): MERGE



Example(cont): CONGRUENCE



Arrays

(1) Quantifier-free Fragment of T_A

$$\Sigma_A : \{ \cdot[\cdot], \cdot \langle \cdot \rangle, = \},$$

where

- ▶ $a[i]$ is a binary function representing read of array a at index i ;
- ▶ $a \langle i \triangleleft v \rangle$ is a ternary function representing write of value v to index i of array a ;
- ▶ $=$ is a binary predicate.

Axioms of T_A :

1. axioms of (reflexivity), (symmetry), and (transitivity) of $T_=$
2. $\forall a, i, j. i = j \rightarrow a[i] = a[j]$ (array congruence)
3. $\forall a, v, i, j. i = j \rightarrow a \langle i \triangleleft v \rangle [j] = v$ (read-over-write 1)
4. $\forall a, v, i, j. i \neq j \rightarrow a \langle i \triangleleft v \rangle [j] = a[j]$ (read-over-write 2)

Note: a may itself be a write term, e.g., $a \langle i' \triangleleft v' \rangle$. Then

$$(a \langle i' \triangleleft v' \rangle) \langle i \triangleleft v \rangle$$

means: first write the value v' to index i' of a
 then write the value v to index i of a

The Decision Procedure

Given quantifier-free conjunctive Σ_A -formula F .

To decide the T_A -satisfiability of F :

Step 1

If F does not contain any write terms $a\langle i \triangleleft v \rangle$, then

1. associate array variables a with fresh function symbol f_a , and replace read terms $a[i]$ with $f_a(i)$;
2. decide the $T_=$ -satisfiability of the resulting formula.

Example: Consider Σ_A -formula

$$F : i_1 = j \wedge i_1 \neq i_2 \wedge a[j] = v_1 \wedge a\langle i_1 \triangleleft v_1 \rangle\langle i_2 \triangleleft v_2 \rangle[j] \neq a[j] .$$

F contains a write term,

$$a\langle i_1 \triangleleft v_1 \rangle\langle i_2 \triangleleft v_2 \rangle[j] \neq a[j] .$$

According to (read-over-write 1), assume $i_2 = j$ and recurse on

$$F_1 : i_2 = j \wedge i_1 = j \wedge i_1 \neq i_2 \wedge a[j] = v_1 \wedge v_2 \neq a[j] .$$

F_1 does not contain any write terms, so rewrite it to

$$F'_1 : i_2 = j \wedge i_1 = j \wedge i_1 \neq i_2 \wedge f_a(j) = v_1 \wedge v_2 \neq f_a(j) .$$

The first two literals imply that $i_1 = i_2$, contradicting the third literal, so F'_1 is $T_=$ -unsatisfiable.

Step 2

Select some read-over-write term $a\langle i \triangleleft v \rangle[j]$ (note that a may itself be a write term) and split on two cases:

1. According to (read-over-write 1), replace

$$F[a\langle i \triangleleft v \rangle[j]] \quad \text{with} \quad F_1 : F[v] \wedge i = j ,$$

and recurse on F_1 . If F_1 is found to be T_A -satisfiable, return satisfiable.

2. According to (read-over-write 2), replace

$$F[a\langle i \triangleleft v \rangle[j]] \quad \text{with} \quad F_2 : F[a[j]] \wedge i \neq j ,$$

and recurse on F_2 . If F_2 is found to be T_A -satisfiable, return satisfiable.

If both F_1 and F_2 are found to be T_A -unsatisfiable, return unsatisfiable.

Returning, we try the second case:

according to (read-over-write 2), assume $i_2 \neq j$ and recurse on

$$F_2 : i_2 \neq j \wedge i_1 = j \wedge i_1 \neq i_2 \wedge a[j] = v_1 \wedge a\langle i_1 \triangleleft v_1 \rangle[j] \neq a[j] .$$

F_2 contains a write term. According to (read-over-write 1), assume $i_1 = j$ and recurse on

$$F_3 : i_1 = j \wedge i_2 \neq j \wedge i_1 = j \wedge i_1 \neq i_2 \wedge a[j] = v_1 \wedge v_1 \neq a[j] .$$

Contradiction because of the final two terms. Thus, according to (read-over-write 2), assume $i_1 \neq j$ and recurse on

$$F_4 : i_1 \neq j \wedge i_2 \neq j \wedge i_1 = j \wedge i_1 \neq i_2 \wedge a[j] = v_1 \wedge a[j] \neq a[j] .$$

Two contradictions: the first and third literals contradict each other, and the final literal is contradictory. As all branches have been tried, F is T_A -unsatisfiable.

Suppose instead that F does not contain the literal $i_1 \neq i_2$. Is this new formula T_A -satisfiable?