The Calculus of Computation: Decision Procedures with Applications to Verification

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10. Combining Decision Procedures

Combining Decision Procedures: Nelson-Oppen Method

Given

Theories T_i over *signatures* Σ_i
(constants, functions, predicates)
with corresponding decision procedures P_i for T_i-satisfiability.

Goal

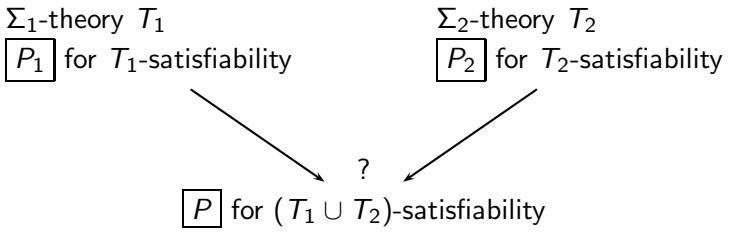
Decide satisfiability of a sentence in theory $\cup_i T_i$.

Example: How do we show that

$$F: \ 1 \leq x \ \wedge \ x \leq 2 \ \wedge \ f(x) \neq f(1) \ \wedge \ f(x) \neq f(2)$$

is $(T_E \cup T_{\mathbb{Z}})$-unsatisfiable?

Combining Decision Procedures



Problem:
Decision procedures are domain specific.
How do we combine them?

Nelson-Oppen Combination Method (N-O Method)

$$\Sigma_1 \cap \Sigma_2 = \emptyset$$

Σ_1-theory T_1
stably infinite

Σ_2-theory T_2
stably infinite

$\boxed{P_1}$ for T_1-satisfiability
of quantifier-free Σ_1-formulae

$\boxed{P_2}$ for T_2-satisfiability
of quantifier-free Σ_2-formulae

$\boxed{P}$ for $(T_1 \cup T_2)$-satisfiability
of quantifier-free $(\Sigma_1 \cup \Sigma_2)$-formulae

Nelson-Oppen: Limitations

Given formula F in theory $T_1 \cup T_2$.

1. F must be quantifier-free.
2. Signatures Σ_i of the combined theory only share $=$, i.e.,

$$\Sigma_1 \cap \Sigma_2 = \{=\}$$

3. Theories must be stably infinite.

Note:

- Algorithm can be extended to combine arbitrary number of theories T_i — combine two, then combine with another, and so on.
- We restrict F to be conjunctive formula — otherwise convert to DNF and check each disjunct.

Stably Infinite Theories

A Σ-theory T is *stably infinite* iff
for every quantifier-free Σ-formula F:
if F is T-satisfiable
then there exists some T-interpretation that satisfies F.

Example: Σ-theory T

$$\Sigma : \{a, b, =\}$$

Axiom

$$\forall x.\ x = a \ \vee \ x = b$$

For every T-interpretation I, $|D_I| \leq 2$ (at most two elements).
Hence, T is *not* stably infinite.

All the other theories mentioned so far are stably infinite.

Example: Theory of partial orders

Σ-theory $T_\preceq$

$$\Sigma_\preceq : \{\preceq, =\}$$

where $\preceq$ is a binary predicate.

Axioms

1. $\forall x.\ x \preceq x$ ($\preceq$ reflexivity)
2. $\forall x, y.\ x \preceq y \ \wedge \ y \preceq x \ \rightarrow \ x = y$ ($\preceq$ antisymmetry)
3. $\forall x, y, z.\ x \preceq y \ \wedge \ y \preceq z \ \rightarrow \ x \preceq z$ ($\preceq$ transitivity)

We prove $T_{\preceq}$ is stably infinite.

Consider $T_{\preceq}$-satisfiable quantifier-free $\Sigma_{\preceq}$-formula F.
Consider arbitrary satisfying $T_{\preceq}$-interpretation $I : (D_I, \alpha_I)$,
where α_I maps $\preceq$ to $\leq_I$.

- Let A be any infinite set disjoint from D_I
- Construct new interpretation $J : (D_J, \alpha_J)$
  - $D_J = D_I \cup A$
  - $\alpha_J = \{\preceq \mapsto \leq_J\}$, where for $a, b \in D_J$,

$$a \leq_J b \stackrel{\text{def}}{=} \begin{cases} a \leq_I b & \text{if } a, b \in D_I \\ a = b & \text{otherwise} \end{cases}$$

J is $T_{\preceq}$-interpretation satisfying F with infinite domain.
Hence, $T_{\preceq}$ is stably infinite.

Example: Consider quantifier-free conjunctive $(\Sigma_E \cup \Sigma_{\mathbb{Z}})$-formula

$$F : \ 1 \leq x \ \wedge \ x \leq 2 \ \wedge \ f(x) \neq f(1) \ \wedge \ f(x) \neq f(2) \ .$$

The signatures of T_E and $T_{\mathbb{Z}}$ only share $=$. Also, both theories are stably infinite. Hence, the NO combination of the decision procedures for T_E and $T_{\mathbb{Z}}$ decides the $(T_E \cup T_{\mathbb{Z}})$-satisfiability of F.

Intuitively, F is $(T_E \cup T_{\mathbb{Z}})$-unsatisfiable.
For the first two literals imply $x = 1 \ \vee \ x = 2$ so that
$f(x) = f(1) \ \vee \ f(x) = f(2)$.
Contradict last two literals.
Hence, F is $(T_E \cup T_{\mathbb{Z}})$-unsatisfiable.

N-O Overview

Phase 1: Variable Abstraction

- Given conjunction Γ in theory $T_1 \cup T_2$.
- Convert to conjunction $\Gamma_1 \cup \Gamma_2$ s.t.
  - Γ_i in theory T_i
  - $\Gamma_1 \cup \Gamma_2$ satisfiable iff Γ satisfiable.

Phase 2: Check

- If there is some set S of equalities and disequalities between the shared variables of Γ_1 and Γ_2
  $\text{shared}(\Gamma_1, \Gamma_2) = \text{free}(\Gamma_1) \cap \text{free}(\Gamma_2)$
  s.t. $S \cup \Gamma_i$ are T_i-satisfiable for all i,
  then Γ is **satisfiable**.
- Otherwise, **unsatisfiable**.

Nelson-Oppen Method: Overview

Consider quantifier-free conjunctive $(\Sigma_1 \cup \Sigma_2)$-formula F.

Two versions:

- nondeterministic — simple to present, but high complexity
- deterministic — efficient

Nelson-Oppen (N-O) method proceeds in two steps:

- Phase 1 (variable abstraction)
  — same for both versions
- Phase 2
  nondeterministic: guess equalities/disequalities and check
  deterministic: generate equalities/disequalities by equality propagation

Phase 1: Variable abstraction

Given quantifier-free conjunctive $(\Sigma_1 \cup \Sigma_2)$-formula F.
Transform F into two quantifier-free conjunctive formulae

Σ_1-formula F_1 and Σ_2-formula F_2

s.t. F is $(T_1 \cup T_2)$-satisfiable iff $F_1 \ \wedge \ F_2$ is $(T_1 \cup T_2)$-satisfiable
F_1 and F_2 are linked via a set of shared variables.

For term t, let $\mathrm{hd}(t)$ be the root symbol, e.g. $\mathrm{hd}(f(x)) = f$.

Generation of F_1 and F_2

For $i, j \in \{1, 2\}$ and $i \neq j$, repeat the transformations

(1) if function $f \in \Sigma_i$ and $\mathrm{hd}(t) \in \Sigma_j$,

$$F[f(t_1, \ldots, t, \ldots, t_n)] \quad \Rightarrow \quad F[f(t_1, \ldots, w, \ldots, t_n)] \wedge w = t$$

(2) if predicate $p \in \Sigma_i$ and $\mathrm{hd}(t) \in \Sigma_j$,

$$F[p(t_1, \ldots, t, \ldots, t_n)] \quad \Rightarrow \quad F[p(t_1, \ldots, w, \ldots, t_n)] \wedge w = t$$

(3) if $\mathrm{hd}(s) \in \Sigma_i$ and $\mathrm{hd}(t) \in \Sigma_j$,

$$F[s = t] \quad \Rightarrow \quad F[\top] \ \wedge \ w = s \ \wedge \ w = t$$

(4) if $\mathrm{hd}(s) \in \Sigma_i$ and $\mathrm{hd}(t) \in \Sigma_j$,

$$F[s \neq t] \quad \Rightarrow \quad F[w_1 \neq w_2] \ \wedge \ w_1 = s \ \wedge \ w_2 = t$$

where w, w_1, and w_2 are fresh variables.

Example: Consider $(\Sigma_E \cup \Sigma_{\mathbb{Z}})$-formula

$$F: \ 1 \leq x \ \wedge \ x \leq 2 \ \wedge \ f(x) \neq f(1) \ \wedge \ f(x) \neq f(2) \ .$$

According to transformation 1, since $f \in \Sigma_E$ and $1 \in \Sigma_{\mathbb{Z}}$, replace $f(1)$ by $f(w_1)$ and add $w_1 = 1$. Similarly, replace $f(2)$ by $f(w_2)$ and add $w_2 = 2$.
Now, the literals

$$\Gamma_{\mathbb{Z}}: \ \{1 \leq x, \ x \leq 2, \ w_1 = 1, \ w_2 = 2\}$$

are $T_{\mathbb{Z}}$-literals, while the literals

$$\Gamma_E: \ \{f(x) \neq f(w_1), \ f(x) \neq f(w_2)\}$$

are T_E-literals. Hence, construct the $\Sigma_{\mathbb{Z}}$-formula

$$F_1: \ 1 \leq x \ \wedge \ x \leq 2 \ \wedge \ w_1 = 1 \ \wedge \ w_2 = 2$$

and the Σ_E-formula

$$F_2: \ f(x) \neq f(w_1) \ \wedge \ f(x) \neq f(w_2) \ .$$

F_1 and F_2 share the variables $\{x, w_1, w_2\}$.
$F_1 \ \wedge \ F_2$ is $(T_E \cup T_{\mathbb{Z}})$-equisatisfiable to F.

Example: Consider $(\Sigma_E \cup \Sigma_{\mathbb{Z}})$-formula

$$F: \ f(x) = x + y \wedge x \leq y + z \wedge x + z \leq y \wedge y = 1 \wedge f(x) \neq f(2) \ .$$

In the first literal, $\mathrm{hd}(f(x)) = f \in \Sigma_{\mathrm{E}}$ and $\mathrm{hd}(x + y) = + \in \Sigma_{\mathbb{Z}}$; thus, by (3), replace the literal with

$$w_1 = f(x) \ \wedge \ w_1 = x + y \ .$$

In the final literal, $f \in \Sigma_E$ but $2 \in \Sigma_{\mathbb{Z}}$, so by (1), replace it with

$$f(x) \neq f(w_2) \ \wedge \ w_2 = 2 \ .$$

Now, separating the literals results in two formulae:

$$F_1: \ w_1 = x + y \ \wedge \ x \leq y + z \ \wedge \ x + z \leq y \ \wedge \ y = 1 \ \wedge \ w_2 = 2$$

is a $\Sigma_{\mathbb{Z}}$-formula, and

$$F_2: \ w_1 = f(x) \ \wedge \ f(x) \neq f(w_2)$$

is a Σ_E-formula.
The conjunction $F_1 \ \wedge \ F_2$ is $(T_E \cup T_{\mathbb{Z}})$-equisatisfiable to F.

Nondeterministic Version

Phase 2: Guess and Check

- Phase 1 separated $(\Sigma_1 \cup \Sigma_2)$-formula F into two formulae:
  Σ_1-formula F_1 and Σ_2-formula F_2
- F_1 and F_2 are linked by a set of shared variables:
  $V = \text{shared}(F_1, F_2) = \text{free}(F_1) \cap \text{free}(F_2)$
- Let E be an equivalence relation over V.
- The arrangement $\alpha(V, E)$ of V induced by E is:

$$\alpha(V,E): \bigwedge_{u,v \in V.\ uEv} u = v \ \wedge \bigwedge_{u,v \in V.\ \neg(uEv)} u \neq v$$

Then,
the original formula F is $(T_1 \cup T_2)$-satisfiable iff
there exists an equivalence relation E of V s.t.
(1) $F_1 \wedge \alpha(V,E)$ is T_1-satisfiable, and
(2) $F_2 \wedge \alpha(V,E)$ is T_2-satisfiable.
Otherwise, F is $(T_1 \cup T_2)$-unsatisfiable.

Example: Consider $(\Sigma_E \cup \Sigma_{\mathbb{Z}})$-formula

$$F: \ 1 \leq x \ \wedge \ x \leq 2 \ \wedge \ f(x) \neq f(1) \ \wedge \ f(x) \neq f(2)$$

Phase 1 separates this formula into the $\Sigma_{\mathbb{Z}}$-formula

$$F_1: \ 1 \leq x \ \wedge \ x \leq 2 \ \wedge \ w_1 = 1 \ \wedge \ w_2 = 2$$

and the Σ_E-formula

$$F_2: \ f(x) \neq f(w_1) \ \wedge \ f(x) \neq f(w_2)$$

with

$$V = \text{shared}(F_1, F_2) = \{x, w_1, w_2\}$$

There are 5 equivalence relations to consider, which we list by stating the partitions:

1. $\{\{x, w_1, w_2\}\}$, *i.e.*, $x = w_1 = w_2$:
   $x = w_1$ and $f(x) \neq f(w_1) \ \Rightarrow \ F_2 \wedge \alpha(V,E)$ is T_E-unsatisfiable.
2. $\{\{x, w_1\}, \{w_2\}\}$, *i.e.*, $x = w_1$, $x \neq w_2$:
   $x = w_1$ and $f(x) \neq f(w_1) \ \Rightarrow \ F_2 \wedge \alpha(V,E)$ is T_E-unsatisfiable.
3. $\{\{x, w_2\}, \{w_1\}\}$, *i.e.*, $x = w_2$, $x \neq w_1$:
   $x = w_2$ and $f(x) \neq f(w_2) \ \Rightarrow \ F_2 \wedge \alpha(V,E)$ is T_E-unsatisfiable.
4. $\{\{x\}, \{w_1, w_2\}\}$, *i.e.*, $x \neq w_1$, $w_1 = w_2$:
   $w_1 = w_2$ and $w_1 = 1 \ \wedge \ w_2 = 2$
   $\Rightarrow \ F_1 \wedge \alpha(V,E)$ is $T_{\mathbb{Z}}$-unsatisfiable.
5. $\{\{x\}, \{w_1\}, \{w_2\}\}$, *i.e.*, $x \neq w_1$, $x \neq w_2$, $w_1 \neq w_2$:
   $x \neq w_1 \ \wedge \ x \neq w_2$ and $x = w_1 = 1 \ \vee \ x = w_2 = 2$
   (since $1 \leq x \leq 2$ implies that $x = 1 \ \vee \ x = 2$ in $T_{\mathbb{Z}}$)
   $\Rightarrow \ F_1 \wedge \alpha(V,E)$ is $T_{\mathbb{Z}}$-unsatisfiable.

Hence, F is $(T_E \cup T_{\mathbb{Z}})$-unsatisfiable.

Example: Consider the $(\Sigma_{\text{cons}} \cup \Sigma_{\mathbb{Z}})$-formula

$$F: \ \text{car}(x) + \text{car}(y) = z \ \wedge \ \text{cons}(x,z) \neq \text{cons}(y,z) \ .$$

After two applications of (1), Phase 1 separates F into the Σ_{cons}-formula

$$F_1: \ w_1 = \text{car}(x) \ \wedge \ w_2 = \text{car}(y) \ \wedge \ \text{cons}(x,z) \neq \text{cons}(y,z)$$

and the $\Sigma_{\mathbb{Z}}$-formula

$$F_2: \ w_1 + w_2 = z \ ,$$

with

$$V = \text{shared}(F_1, F_2) = \{z, w_1, w_2\} \ .$$

Consider the equivalence relation E given by the partition

$$\{\{z\}, \{w_1\}, \{w_2\}\} \ .$$

The arrangement

$$\alpha(V,E): \ z \neq w_1 \ \wedge \ z \neq w_2 \ \wedge \ w_1 \neq w_2$$

satisfies both F_1 and F_2: $F_1 \wedge \alpha(V,E)$ is T_{cons}-satisfiable, and $F_2 \wedge \alpha(V,E)$ is $T_{\mathbb{Z}}$-satisfiable.
Hence, F is $(T_{\text{cons}} \cup T_{\mathbb{Z}})$-satisfiable.

Practical Efficiency

Phase 2 was formulated as "guess and check":
First, guess an equivalence relation E,
then check the induced arrangement.

The number of equivalence relations grows super-exponentially with the # of shared variables. It is given by Bell numbers.
e.g., 12 shared variables $\Rightarrow$ over four million equivalence relations.

Solution: Deterministic Version

Deterministic Version

Phase 1 as before
Phase 2 asks the decision procedures P_1 and P_2 to propagate new equalities.

Example 1:

Real linear arithmethic $T_{\mathbb{R}}$ $\boxed{P_{\mathbb{R}}}$ Theory of equality T_E $\boxed{P_E}$

$$F: \quad f(f(x)-f(y)) \neq f(z) \ \wedge \ x \leq y \ \wedge \ y+z \leq x \ \wedge \ 0 \leq z$$

$(T_{\mathbb{R}} \cup T_E)$-unsatisfiable

Intuitively,
last 3 conjuncts $\Rightarrow$ $x = y \ \wedge \ z = 0$
contradicts 1st conjunct

Phase 1: Variable Abstraction

$$F: \ f(f(x)-f(y)) \neq f(z) \ \wedge \ x \leq y \ \wedge \ y+z \leq x \ \wedge \ 0 \leq z$$

$$f(x) \ \Rightarrow \ u \qquad f(y) \ \Rightarrow \ v \qquad u-v \ \Rightarrow \ w$$

Γ_E: $\{f(w) \neq f(z), \ u = f(x), \ v = f(y)\}$ ... T_E-formula

$\Gamma_{\mathbb{R}}$: $\{x \leq y, \ y+z \leq x, \ 0 \leq z, \ w = u-v\}$ ... $T_{\mathbb{R}}$-formula

$$\text{shared}(\Gamma_{\mathbb{R}}, \Gamma_E) = \{x, y, z, u, v, w\}$$

Nondeterministic version — over 200 Es!
Let's try the deterministic version.

Phase 2: Equality Propagation

$\boxed{P_{\mathbb{R}}}$ $\Gamma_{\mathbb{R}} \models x = y$

$s_0 : \langle \Gamma_{\mathbb{R}}, \Gamma_E, \{\} \rangle$ $\boxed{P_E}$

$s_1 : \langle \Gamma_{\mathbb{R}}, \Gamma_E, \{x = y\} \rangle$ $\Gamma_E \cup \{x = y\} \models u = v$

$s_2 : \langle \Gamma_{\mathbb{R}}, \Gamma_E, \{x = y, u = v\} \rangle$ $\Gamma_{\mathbb{R}} \cup \{u = v\} \models z = w$

$s_3 : \langle \Gamma_{\mathbb{R}}, \Gamma_E, \{x = y, u = v, z = w\} \rangle$ $\Gamma_E \cup \{z = w\} \models \text{false}$

$s_4 : \text{false}$

Contradiction. Thus, F is $(T_{\mathbb{R}} \cup T_E)$-unsatisfiable.
If there were no contradiction, F would be $(T_{\mathbb{R}} \cup T_E)$-satisfiable.

Convex Theories

Claim:
Equality propagation is a decision procedure for convex theories.

Def. A Σ-theory T is *convex* iff
for every quantifier-free conjunction Σ-formula F
and for every disjunction $\bigvee_{i=1}^{n}(u_i = v_i)$

if $F \models \bigvee_{i=1}^{n}(u_i = v_i)$

then $F \models u_i = v_i$, for some $i \in \{1, \ldots, n\}$

Convex Theories

- T_E, $T_{\mathbb{R}}$, $T_{\mathbb{Q}}$, T_{cons} are convex
- $T_{\mathbb{Z}}$, T_{A} are not convex

Example: $T_{\mathbb{Z}}$ is not convex

Consider quantifier-free conjunctive

$$F: \quad 1 \leq z \ \wedge \ z \leq 2 \ \wedge \ u = 1 \ \wedge \ v = 2$$

Then

$$F \models z = u \vee z = v$$

but

$$F \not\models z = u$$
$$F \not\models z = v$$

Example:

The theory of arrays T_{A} is not convex.
Consider the quantifier-free conjunctive Σ_{A}-formula

$$F: \ a\langle i \triangleleft v\rangle[j] = v \ .$$

Then

$$F \ \Rightarrow \ i = j \ \vee \ a[j] = v \ ,$$

but

$$F \not\Rightarrow i = j$$
$$F \not\Rightarrow a[j] = v \ .$$

What if T is Not Convex?

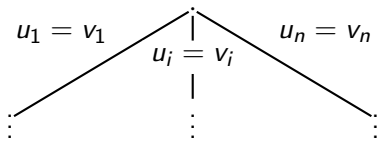
Case split when:

$$\Gamma \models \bigvee_{i=1}^{n}(u_i = v_i)$$

but

$$\Gamma \not\models u_i = v_i \qquad \text{for all } i = 1, \ldots, n$$

- For each $i = 1, \ldots, n$, construct a branch on which $u_i = v_i$ is assumed.
- If all branches are contradictory, then **unsatisfiable**. Otherwise, **satisfiable**.



Example 2: Non-Convex Theory

$T_{\mathbb{Z}}$ not convex! $P_{\mathbb{Z}}$ $\quad$ T_E convex P_E

$$\Gamma : \left\{ \begin{array}{ll} 1 \leq x, & x \leq 2, \\ f(x) \neq f(1), & f(x) \neq f(2) \end{array} \right\} \quad \text{in } T_{\mathbb{Z}} \cup T_E$$

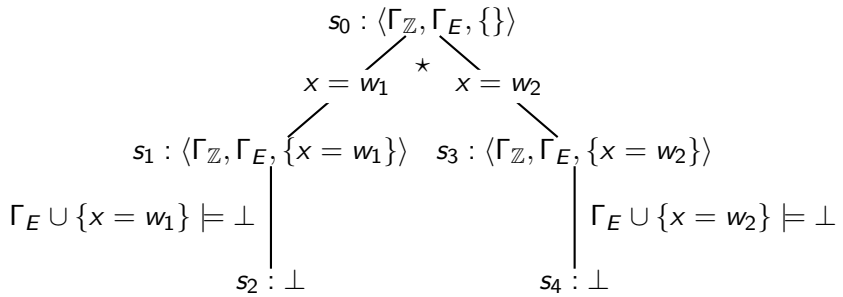
- Replace $f(1)$ by $f(w_1)$, and add $w_1 = 1$.
- Replace $f(2)$ by $f(w_2)$, and add $w_2 = 2$.

Result:

$$\Gamma_{\mathbb{Z}} = \left\{ \begin{array}{l} 1 \leq x, \\ x \leq 2, \\ w_1 = 1, \\ w_2 = 2 \end{array} \right\} \quad \text{and} \quad \Gamma_E = \left\{ \begin{array}{l} f(x) \neq f(w_1), \\ f(x) \neq f(w_2) \end{array} \right\}$$

$\text{shared}(\Gamma_{\mathbb{Z}}, \Gamma_E) = \{x, w_1, w_2\}$

Example 2: Non-Convex Theory



$\star$: $\Gamma_{\mathbb{Z}} \models x = w_1 \ \vee \ x = w_2$

All leaves are labeled with $\bot \Rightarrow \Gamma$ is $(T_{\mathbb{Z}} \cup T_E)$-unsatisfiable.

Example 3: Non-Convex Theory

$$\Gamma : \left\{ \begin{array}{lll} 1 \leq x, & x \leq 3, & \\ f(x) \neq f(1), & f(x) \neq f(3), & f(1) \neq f(2) \end{array} \right\} \quad \text{in } T_{\mathbb{Z}} \cup T_E$$

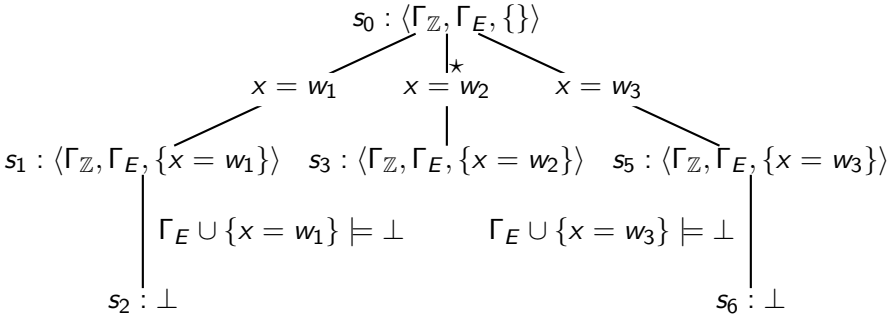
- Replace $f(1)$ by $f(w_1)$, and add $w_1 = 1$.
- Replace $f(2)$ by $f(w_2)$, and add $w_2 = 2$.
- Replace $f(3)$ by $f(w_3)$, and add $w_3 = 3$.

Result:

$$\Gamma_{\mathbb{Z}} = \left\{ \begin{array}{l} 1 \leq x, \\ x \leq 3, \\ w_1 = 1, \\ w_2 = 2, \\ w_3 = 3 \end{array} \right\} \quad \text{and} \quad \Gamma_E = \left\{ \begin{array}{l} f(x) \neq f(w_1), \\ f(x) \neq f(w_3), \\ f(w_1) \neq f(w_2) \end{array} \right\}$$

$\text{shared}(\Gamma_{\mathbb{Z}}, \Gamma_E) = \{x, w_1, w_2, w_3\}$

Example 3: Non-Convex Theory



$\star$: $\Gamma_{\mathbb{Z}} \models x = w_1 \ \vee \ x = w_2 \ \vee \ x = w_3$

No more equations on middle leaf $\Rightarrow \Gamma$ is $(T_{\mathbb{Z}} \cup T_E)$-satisfiable.