CS 369P: Polyhedral techniques in combinatorial optimization
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1 Bounded Degree Spanning Trees

In this lecture we try to solve the following problem:

Problem 1 *Given an undirected graph $G = (V, E)$, with edge weights w_e for each $e \in E$, and a degree bound $B_v \geq 1$ for each $v \in V$. The goal is to find a spanning tree of minimum weight such that for any vertex v, the degree of v in T is at most B_v:*

$$\forall v \in V;\ deg_T(v) \leq B_v \tag{1}$$

In general, even when all of the weights are equal it is NP-hard to decide wether there is a spanning tree that satisfies equation (1). In particular, suppose $B_v = 2$ for all except two of the vertices. In this case problem 1 is equivalent to finding a hamiltonian path in G, which is an NP-hard problem.

To establish a linear programming relaxation of the problem, we use the spanning tree polytope. Recall that the spanning tree polytope (and more generally the matroid polytope) is:

$$P_{s.t.}(G) = \{\mathbf{x} \in \mathbb{R}_+^E : \ \forall S \subseteq E; x(S) \leq r(S),\ x(E) = r(E)\}, \tag{2}$$

where $r(S) = |V| - \#components(S)$ is the rank function, and $\#components(S)$ is the number of connected components of the subgraph (V, S). For example $r(E) = |V| - 1$.

A natural LP relaxation of the problem may be obtained simply by adding the degree constraints to the spanning tree polytope:

$$\text{minimize} \quad \sum_{e \in E} w_e x_e : \tag{3}$$
$$\sum_{e \in E} x_e = |V| - 1,$$
$$\forall S \subseteq E; \quad \sum_{e \in S} x_e \leq r(S), \tag{4}$$
$$\forall v \in V; \quad \sum_{e \in \delta(v)} x_e \leq B_v, \tag{5}$$
$$x_e \geq 0.$$

Let us define some notations. Let $\mathbf{x} : E \to \mathbb{R}$ be a vertex solution of LP (3).

Definition 1 *We say a set $S \subseteq E$ is tight with respect to $\mathbf{x}$, if equation (4) is satisfied with equality for S (i.e $x(S) = r(S)$). Similarly, we say a vertex $v \in V$ is tight with respect to $\mathbf{x}$, if equation (5) is satisfied with equality for v (i.e. $x(\delta(v)) = B_v$).*

We start by describing the properties of vertex solution $\mathbf{x}$. Recall that for any set S, $\chi_S : E \rightarrow \{0,1\}$ is the characteristic vector of S such that $\chi_S(e) = 1$ iff $e \in S$. By the analysis of matroid polytopes, we have the following lemma:

Lemma 2 *For all $\mathbf{x} \in P_{s.t.}(G)$, there exists a chain of tight sets $C_1 \subset C_2 \subset \ldots \subset C_k \subseteq E$ such that for every tight set T we have $\chi_T \in span(\{\chi_{C_i} : 1 \leq i \leq k\})$.*

Similar to the proofs in previous lectures, we establish some basic properties of the vertex solutions. Then, we gradually relax the LP (3) by removing constraints, and hopefully preserving the properties of the optimum solution. For this problem we gradually remove the degree constraints for the vertices until all of them is removed. Thus we need to work with a a relaxation of LP (3). Let $\tilde{V} \subset V$ be the set of vertices that still has the degree constraint. Define:

$$P(\tilde{V}) := \{\mathbf{x} \in \mathbb{R}_+^E : x(E) = |V| - 1,\ \forall S \subset E; x(S) \leq r(S),\ \forall v \in \tilde{V}; x(\delta(v)) \leq B_v\}$$

We start the algorithm with $\tilde{V} = V$, and we gradually remove vertices from $\tilde{V}$ until $\tilde{V} = \emptyset$.

Before describing the algorithm we need to know more properties of $\mathbf{x}$. The following lemma which is an immediate consequence of the Lemma 2 describes more properties of $\mathbf{x}$.

Lemma 3 *Let $\tilde{V} \subset V$. If $\mathbf{x}$ is a vertex of $P(\tilde{V})$, then there is a chain of tight sets $\mathcal{C} \subseteq 2^E$, and a set of tight sets $\Gamma \subset \tilde{V}$ such that $|\mathcal{C}| + |\Gamma| = |supp(\mathbf{x})|$, and the corresponding vectors $\{\chi_C : C \in \mathcal{C}\}$ and $\{\chi_{\delta(v)} : v \in \Gamma\}$ are linearly independent (recall that $supp(\mathbf{x}) := \{e : x_e > 0\}$ is the set of non-zero elements of vector $\mathbf{x}$).*

Proof: By Lemma 2 we can choose a chain of tight sets $\mathcal{C}'$ such that for every tight set T, we have $\chi_T \in span(\{\chi_C : C \in \mathcal{C}\}$. Also let $\Gamma' \subseteq \tilde{V}$ be the set of tight vertices w.r.t. $\mathbf{x}$. By the 4^{th} definition of a vertex of a polytope, there must be $|E|$ linearly independent tight constraint for $\mathbf{x}$, where $|supp(x)|$ of which must correspond to tight sets and tight vertices. Therefore, we can choose a set $\mathcal{C} \subset \mathcal{C}'$ of tight sets and $\Gamma \subset \Gamma'$ of tight vertices such that $|\mathcal{C}| + |\Gamma| = |supp(x)|$, and $\mathcal{C}$ is a chain. The latter follows from the fact that any subset of a chain is also a chain. □

As we said above, we gradually remove the degree constraints in LP (3). But we need to be sure that the degree of the vertex in the final spanning tree remains close to its bound. Hence, we only allow removing the vertices v that have at most one edge more than B_v. In the following lemma which is the most crucial part of the proof we show that we can always find such a vertex and remove it from $\tilde{V}$:

Lemma 4 *Let $\mathbf{x}$ be a vertex of $P(\tilde{V})$, and $E_+ = supp(x)$. If $\tilde{V} \neq \emptyset$, then there exists a vertex $v \in \tilde{V}$ such that $deg_{E_+}(v) \leq B_v + 1$, where $deg_{E_+}(v)$ is the degree of v in the subgraph (V, E_+).*

Proof: The proof is by contradiction. Assume wlog that $E = supp(x)$ (remove the zero edges from the graph), and for all vertex $v \in \tilde{V}$:

$$deg(v) \geq B_v + 2. \tag{6}$$

Also let $\mathcal{C} = C_1 \subset C_2 \subset \ldots \subset C_k$, and Γ be the chain of tight sets and tight vertices described in Lemma 3. We use a double counting argument. We give \$1 to each edge $e \in E$ and then we distribute the money among the tight sets and vertices in $\mathcal{C}$ and V such that each set $C_i \in \mathcal{C}$, and each vertex $v \in \tilde{V}$ receives at least \$1. Since we start with $|E| = |\mathcal{C}| + |\Gamma|$ dollars this is possible only if each tight set in $\mathcal{C}$ and each vertex in Γ receives exactly \$1, $\Gamma = \tilde{V}$, $C_k = E$ and each $v \in V \setminus \tilde{V}$ receives nothing. Finally, we reach to a contradiction by showing that this implies the characteristic vectors of $\chi_{\delta(v)}$ for $v \in \Gamma$ are not linearly independent of χ_{C_i} for $C_i \in \mathcal{C}$.

We distribute the money as follows: For an edge $e \in E$, we give x_e to the smallest $C_i \in \mathcal{C}$ such that $C_i \ni e$. We split the rest of the money between the two neighbours of e. In particular if $e = (u, v)$ we give $\frac{1}{2}(1 - x_e)$ to each of u and v. Let $r(C_i)$, and $r(v)$ be the total amount received by tight set $C_i \in \mathcal{C}$ and $v \in \tilde{V}$ respectively. We show that $r(C_i) \geq 1$, and $r(v) \geq 1$.

First we show that for each set $C_i \in \mathcal{C}$, $r(C_i) \geq 1$. Let $C_i \in \mathcal{C}$ be a tight set. Observe that for each edge $e \in C_i \setminus C_{i-1}$, C_i receives x_e (wlog assume $C_0 = \emptyset$). Therefore:

$$r(C_i) = \sum_{e \in C_i \setminus C_{i-1}} x_e = x(C_i) - x(C_{i-1}) \geq 1, \tag{7}$$

where the last inequality follows from the fact that for all $e \in C_i \setminus C_{i-1}$, $x_e > 0$ and for all tight sets T, $x(T) = r(T) = |V| - \#\mathit{components}(T)$ is an integer. Next we show that for each vertex $v \in \tilde{V}$, $r(v) \geq 1$. Let $v \in \tilde{V}$; since it receives $\frac{1}{2}(1 - x_e)$ from each of its incident edges we have:

$$r(v) = \sum_{e \in \delta(v)} \frac{1}{2}(1 - x_e) = \frac{1}{2} deg(v) - \frac{1}{2} x(\delta(v)) \geq \frac{1}{2}(B_v + 2) - \frac{1}{2} B_v = 1, \tag{8}$$

where the last inequality follows from equations (5) and (6). Summing up equations (7) and (8) over all $v \in \tilde{V}$ and $\mathcal{C}_i \in C$ we collect at least

$$\sum_{C_i \in \mathcal{C}} r(C_i) + \sum_{v \in \tilde{V}} r(v) \geq |\mathcal{C}| + |\tilde{V}| \geq |\mathcal{C}| + |\Gamma| = |E|.$$

Since we start the process with only $|E|$ dollars, all of the above equations must be tight. In particular, we obtain:

1. For all $C_i \in \mathcal{C}$ we must have $x(C_i) - x(C_{i-1}) = 1$,
2. Since for all $e \in E$, $x_e > 0$, all edges must be covered by at least one $C_i \in \mathcal{C}$ (i.e. $C_k = E$),
3. For all vertices in $\tilde{V}$ we must have $r(v) = 1$, thus all vertices are tight, and $\Gamma = \tilde{V}$,
4. No other vertices receive any money (i.e. if $v \notin \tilde{V}$, and $e \in \delta(v)$ we must have $x_e = 1$).

Let $\mathbf{v} := 2\chi_E$. We can write $\mathbf{v}$ as follows:

$$\mathbf{v} = 2\chi_E = \sum_{v \in V} \chi_{\delta(v)} = \sum_{v \in \tilde{V}} \chi_{\delta(v)} + \sum_{v \in V \setminus \tilde{V}} \sum_{e \in \delta(v)} \chi_e. \tag{9}$$

Let $\mathbf{v}_1 := \sum_{v \in \tilde{V}} \chi_{\delta(v)}$, and $\mathbf{v}_2 := \sum_{v \in V \setminus \tilde{V}} \chi_{\delta(v)}$ be the first and second vector in the RHS of the above equation, respectively. Also let $\mathbf{V}_{\mathcal{C}} := \mathit{span}\{\chi_{C_i} : C_i \in \mathcal{C}\}$, $\mathbf{V}_{\tilde{V}} := \mathit{span}\{\chi_{\delta(v)} : v \in \tilde{V}\}$. We show that $\mathbf{v}_1 \in \mathbf{V}_{\tilde{V}}$, and $\mathbf{v}_2 \in \mathbf{V}_{\mathcal{C}}$. Since $C_k = E$, we also have $2\chi_E \in \mathbf{V}_{\mathcal{C}}$; hence we have $\mathbf{v}_1 = \mathbf{v} - \mathbf{v}_2$ must be in both $\mathbf{V}_{\tilde{V}}$ and $\mathbf{V}_{\mathcal{C}}$. Since $B_v \geq 0$, by equation (6) for any vertex $v \in \tilde{V}$, we have $\delta(v) \neq \emptyset$; thus $\mathbf{v}_1 \neq 0$. Therefore, the set of vectors $\{\chi_{\delta(v)} : v \in \tilde{V} = \Gamma\}$ and $\{\chi_{C_i} : C_i \in \mathcal{C}\}$ are linearly independent which by Lemma 3 is a contradiction.

It remains to show that $\mathbf{v}_1 \in \mathbf{V}_{\tilde{V}}$ and $\mathbf{v}_2 \in \mathbf{V}_{\mathcal{C}}$. Since $\Gamma = \tilde{V}$ and $\mathbf{v}_1$ is just the summation of vectors $\chi_{\delta(v)}$ for $v \in \tilde{V}$ we have $\mathbf{v}_1 \in \mathbf{V}_{\tilde{V}}$. Let $e \in \delta(v)$ where $v \in V \setminus \tilde{V}$, Since $x_e = 1$, the whole \$1 is given to a set $C_i \in \mathcal{C}$. However, since C_i received exactly \$1 (i.e. $x(C_i) - x(C_{i-1}) = 1$), we must have $C_i = C_{i-1} + e$. Therefore, for any e adjacent to a vertex $v \in V \setminus \tilde{V}$ we have $\chi_e = \chi_{C_i} - \chi_{C_{i-1}}$ for some $C_i \in \mathcal{C}$. Therefore, $v_2 \in \mathbf{V}_{\mathcal{C}}$, and $\mathbf{v}_1 \in \mathbf{V}_{\mathcal{C}}, \mathbf{V}_{\tilde{V}}$ which is a contradiction. □

Algorithm 1 describe the details of the final algoritm.

Algorithm 1 Algorithm for Bounded Degree Spanning Tree Problem

Input: Weighted graph $G = (V, E)$ with cost function $w : E \rightarrow R$, and degree bounds $B_v : V \rightarrow \mathbb{N}$.
Output: A spanning tree T of minimum weight such that for any vertex v, $deg_T(v) \leq B_v + 1$.

Initialize $\tilde{V} \leftarrow V$.
while $\tilde{V} \neq \emptyset$ **do**
  Solve LP, find a vertex solution $\mathbf{x}$.
  Remove all edges $e \in E$ with $x_e = 0$
  Find a vertex $v \in \tilde{V}$ such that $deg(v) \leq B_v + 1$; set $\tilde{V} \leftarrow \tilde{V} - v$. {remove the degree constraint of v.}
end while
return minimum spanning tree in the final subgraph (V, E). {Since $\tilde{V} = \emptyset$ there is no degree constraint left, and the LP is just the spanning tree LP where we can optimize in polynomial time.}

The algorithm trivially terminates after running the loop for n iterations. The following claim finishes the proof of the correctness of the algorithm:

Claim 5 *If there is a spanning tree T^* satisfying $deg_{T^*}(v) \leq B_v$ for all $v \in V$ of cost C, then algorithm 1 finds a spanning tree T of cost at most C such that $deg_T(v) \leq B_v + 1$ for all $v \in V$.*

Proof: Since we remove a degree constraint only if $deg(v) \leq B_v + 1$, we cannot violate the degree constraint of v by more than one; thus $deg_T(v) \leq B_v + 1$. Moreover, since we always relax the Linear Program by removing some of the constraints the feasible solutions of $P(V)$ is also a feasible solution of $P(\emptyset)$. Therefore, the value of LP never increases in the process. □

The result can be simply generalized to any matroids:

Theorem 6 *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. and $H = (E, \mathcal{F})$ be a hyper-graph encoding the following constraints: for all $F \in \mathcal{F}$, $|F \cap B| \leq B_F$. Then there is a polynomial time algorithm that can find a base B of $\mathcal{M}$ of optimal cost such that for all $F \in \mathcal{F}$ satisfies $|F \cap B| \leq B_F + d - 1$, where d is the degree of an element in the hyper-graph H.*