CS 369P: Polyhedral techniques in combinatorial optimization
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1 Spanning Trees

Given a graph $G = (V, E)$ and edge weights $w_e \geq 0$, our goal is to connect all vertices by a subset of edges F while minimizing its cost $\sum_{e \in F} w_e$. Without loss of generality the optimal solution is a tree which is called the Minimum Spanning Tree (MST). This is perhaps the oldest combinatorial optimization problem; it was first solved by Borůvka in 1926 and Jarník in 1930. Both of the proposed algorithms were variants of the greedy algorithm.

Algorithm 1 Borůvka's Algorithm

```
F ← ∅
while F is disconnected do
  for all components C_i do
    F ← F ∪ {e_i} for e_i = the min-weight edge leaving C_i.
  end for
end while
```

Algorithm 2 Jarník's Algorithm

```
T ← ∅
while T is not a spanning tree do
  T ← T ∪ {e} for e = the min-weight edge extending the tree T to a new vertex.
end while
```

In the 1950's the problem was studied again and algorithms were proposed by Prim, Kruskal, and Dijkstra.

Algorithm 3 Kruskal's Algorithm

```
S ← E
F ← ∅
while S ≠ ∅ and F is not spanning do
  Remove the min-weight edge e from S.
  if F ∪ {e} does not create a cycle then
    F ← F ∪ {e}
  else
    Discard e
  end if
end while
```

All of these algorithms work because spanning trees form a "matroid".

2 Matroids

Definition 1 *$\mathcal{M} = (E, \mathcal{I})$, where $\emptyset \neq \mathcal{I} \subseteq 2^E$ is a matroid if*

1. *$\forall I \in \mathcal{I}$, $J \subset I \Rightarrow J \in \mathcal{I}$ ("down-closed family").*
2. *$\forall I, J \in \mathcal{I}$, $|I| < |J| \Rightarrow \exists j \in J \setminus I$ such that $I \cup \{j\} \in \mathcal{I}$ ("extension axiom").*

The above definition captures many interesting notions of "independence", including linear independence among vectors, as we will see below.

2.1 Notation and Terminology

- $I + j \equiv I \cup \{j\}$
- $I - j \equiv I \setminus \{j\}$
- I is an independent set in $\mathcal{M} \iff I \in \mathcal{I}$.
- B is a base in $\mathcal{M} \iff B \in \mathcal{I}$ and B cannot be extended to a larger independent set. More generally, B is a base of $S \subseteq E$, if $B \in \mathcal{I}$ and $\nexists x \in S \setminus B$ such that $B + x \in \mathcal{I}$.

Lemma 2 *$\mathcal{M} = (E, \mathcal{I})$ is a matroid $\iff$ $\mathcal{I}$ is down-closed and $\forall S \subseteq E$ all bases of S have the same size.*

Proof:

1. If $\mathcal{M}$ is a matroid and B_1, B_2 are bases of $S \subseteq E$, then if $|B_1| < |B_2| \implies B_1$ can be extended by $x \in B_2 \setminus B_2$, hence B_1 was not a base.
2. Assume that $\forall S \subseteq E$ all bases have the same size. Then for $I, J \subseteq E$ where $|I| < |J|$ let $S = I \cup J$. Then I is not a base of S, therefore $\exists j \in S \setminus I = J \setminus I$ such that $I + j \in \mathcal{I}$.

□

2.2 Graphic Matroids

So why are spanning trees an example of a matroid?

Definition 3 *For a graph $G = (V, E)$ a forest is any set of edges $F \subseteq E$ that does not contain any cycles.*

Lemma 4 *$\mathcal{M} = (E, \mathcal{F})$ where $\mathcal{F} = \{F \subseteq E : F \text{ is a forest}\}$ is a matroid.*

Proof: $\mathcal{F}$ is clearly down-closed. We prove that for any $S \subseteq E$, all bases have the same size. Take any $S \subseteq E$ and look at the connected components of (V, S). What are the bases of S, i.e. maximal subsets $F \subseteq S$ which are forests? In each connected component C_i we have $|F \cap E[C_i]| = |C_i| - 1$, because if we have less than $|C_i| - 1$ edges, then F does not contain a spanning tree on C_i and we can add some edge without creating a cycle. Conversely, we cannot have more than $|C_i| - 1$ edges on C_i because we would create a cycle. This implies that $|F| = \sum |C_i| - 1 = |V| - \#\text{components}(S)$, the same number for each base F. □

2.3 The Rank Function

Definition 5 *The rank function of a matroid* $\mathcal{M} = (E, \mathcal{I})$ *is* $r(S) = max\{|I|: I \in \mathcal{I}, I \subseteq S\}$.

Recall that all bases of S have the same size, i.e. the rank function says what this size is. This is identical to the notion of dimension in vector spaces.

Definition 6 *A matroid* $\mathcal{M} = (E, \mathcal{I})$ *is graphic if* $\mathcal{I} = \{F \subseteq E : F \text{ is acyclic}\}$ *for some graph* $G = (V, E)$.

Remark 7 *In graphic matroids* $r(S) = |V| - \#components(S)$.

2.4 Other Examples of Matroids

1. Linear Matroid: $\mathcal{M} = (E, \mathcal{I})$ where $E \subseteq \mathbb{V}$ for some vector space $\mathbb{V}$ and $I \in \mathcal{I} \iff I$ is linearly independent in $\mathbb{V}$.

2. Uniform Matroid: $\mathcal{I} = \{I : |I| \leq k\}$.

3. Partition Matroid: $E = E_1 \cup E_2 \cup \ldots \cup E_\ell$, a disjoint union, and $k_1, k_2, \ldots, k_l \in \mathbb{Z}$ such that $\mathcal{I} = \{I \subseteq E : \forall j; |I \cap E_j| \leq k_j\}$.

4. Transversal Matroid: Given $X_1, X_2, \ldots, X_\ell \subseteq E$, not neccessarily disjoint,

$$\begin{aligned} \mathcal{I} &= \{T \subseteq E : T \text{ is a partial transversal of } X_1, X_2, \ldots, X_\ell\} \\ &= \{T \subseteq E : T = \{t_1, \ldots, t_k\}, t_j \in X_{i_j} \text{ where } i_1, \ldots, i_k \text{ are distinct}\} \end{aligned}$$

5. Matching Matroid: (generalizes transversal matroids) Given a graph $G = (V, E)$, let $\mathcal{M} = (V, \mathcal{I})$ where $\mathcal{I} = \{I \subseteq V : I \text{ is covered by some matching in } G\}$.

6. Gammoid: (generalizes transversal matroids) Given a directed graph $D = (V, A)$ and two sets of vertices $S, T \subseteq V$, let $\mathcal{M} = (T, \mathcal{I})$ where $\mathcal{I} = \{J \subseteq T : \exists I \subseteq S, |I| = |J|$, and there are $|I|$ vertex-disjoint paths from I to $J\}$.

7. Deltoid: (special case of a transversal matroid) Given a bipartite graph $G = (V_1 \cup V_2, E)$, let $\mathcal{M}$ be the transversal matroid defined by the set system $X_v = \{v\} \cup N(v)$ for each $v \in V_1$.

   Equivalently, one can define the bases of $\mathcal{M}$ as $B = V_1 \Delta A$ where A are the vertices of some matching in G.

All the matroids above are special cases of linear matroids, i.e. can be represented by linear independence among vectors in some vector space.