MATH 108: Introduction to Combinatorics, Winter 2016
HOMEWORK 6
Due Monday, February 29

You should solve the homework on your own. Don't use any books or the internet.

Problem 1. Let $b_{k,n}$ denote the number of ways one can place n identical balls into k labeled boxes. For a fixed $k \geq 1$, define a generating function $B_k(x) = \sum_{n=0}^{\infty} b_{k,n} x^n$.

- Find a simple formula for $B_1(x)$. (This is a trivial counting problem.)
- Find a simple formula for $B_k(x)$. (Remember that multiplication of generating functions corresponds to combinations of configurations.)
- Determine $b_{k,n}$ by expanding $B_k(x)$ as a power series.
- Is there a direct combinatorial argument to obtain the same answer?

Problem 2. Prove by suitable bijections that the following numbers are equal:

- The number of sequences $\sigma \in \{+1, -1\}^{2n}$ such that $\sum_{i=1}^{k} \sigma_i \geq 0$ for every $1 \leq k \leq 2n$, and $\sum_{i=1}^{2n} \sigma_i = 0$.
- The number of ways to arrange the numbers $\{1, 2, \ldots, 2n\}$ in a $2 \times n$ array so that each row and each column is increasing.
- The number of paths from $(0, 0)$ to (n, n), where the steps are in the direction of either $(+1, 0)$ or $(0, +1)$, and the path must never drop below the diagonal connecting $(0, 0)$ and (n, n).

What are these numbers called?

Problem 3. Find a "de Bruijn" sequence for permutations of 4 things. That is, an arrangement of $1, 2, ..., 24$ such that if a window of length 4 is run along (including around the corner) the relative order of the 4 elements under the window runs through all 24 permutations once and only once. Hint, see pblm. 111 on pg. 354.

Problem 4. Let H be a subgroup of the symmetric group $\mathbb{S}_n$. Show that there is a set of permutations X such that the left cosets aH for $a \in X$ are disjoint, their union is $\mathbb{S}_n$, and the same holds for the right cosets $Ha, a \in X$.
Hint: remember Hall's marriage theorem.