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Math 233: Non-constructive methods in combinatorics | **Instructor: Jan Vondrák**
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Lecture 2. The Lovász Local Lemma

2.1 Introduction and motivation

We start with the *Lovász Local Lemma*, a fundamental tool of the "probabilistic method" and a prototypical non-constructive argument in combinatorics — proving that a certain object exists without showing what it looks like. Often in applying the probabilistic method, one is trying to show that it is possible to avoid "bad events" $\mathcal{E}_1, \dots, \mathcal{E}_n$ with positive probability, or in other words,

$$\mathbb{P}\left[\bigcap_{i=1}^{n} \overline{\mathcal{E}}_i\right] > 0.$$

Here $\mathcal{E}_i$ are subsets of a probability space Ω (typically a finite set), and $\overline{\mathcal{E}}_i = \Omega \setminus \mathcal{E}_i$ denotes the complementary event for each i.

If $\sum_i \mathbb{P}[\mathcal{E}_i] < 1$, then the above inequality clearly follows, by the "union bound". However, this is often not a strong enough tool, since the sum $\sum_i \mathbb{P}[\mathcal{E}_i]$ may be much larger than 1 even if the events can be avoided.

A weaker constraint on the individual probabilities $\mathbb{P}[\mathcal{E}_i]$ is sufficient if the events $\mathcal{E}_i$ are also independent. In that case if $\mathbb{P}[\mathcal{E}_i] < 1$ for all i, then $\mathbb{P}[\cap_i \overline{\mathcal{E}}_i]$ is clearly positive. The Lovász Local Lemma is an effective refinement of this phenomenon, for events that do not have "too much (inter)dependency" – a notion that will be made precise presently. An additional attractive feature of the Local Lemma is that it does not place any restriction on the (finite) number of events $\mathcal{E}_i$.

2.2 Symmetric Local Lemma and application to hypergraph colorability

Before we state and prove the Local Lemma, we first present a prototypical application of the result, which serves to motivate it.

Example 2.1 (Hypergraph 2-coloring) *Given an integer $k \geq 2$, a* k-uniform hypergraph $G = (V(G), E(G))$ *consists of a finite set of nodes $V(G)$ and a collection of subsets $e_1, \dots, e_n \subset V(G)$, each of size k, which are termed "edges" (or "hyper-edges"). We want to color each node in $V(G)$ either red or blue. Under what conditions can we guarantee that there is a coloring with no monochromatic edge, i.e., every edge contains both red and blue nodes? Such hypergraphs are said to be* 2-colorable.

Notice, if we color each node red or blue uniformly at random, then the event $\mathcal{E}_i$ that the ith edge is monochromatic has probability 2^{1-k}. Thus if the hypergraph G has less than 2^{k-1} edges, then by the union bound, the probability that there is at least one monochromatic edge is $< 2^{k-1} \cdot 2^{1-k} = 1$. It follows that G is 2-colorable.

However, this argument fails when G has $\geq 2^{k-1}$ edges. In this case, under what assumptions can we prove 2-colorability? One such assumption is that every edge intersects at most d other edges, for some d. Under such an assumption, we will show 2-colorability using the Lovász Local Lemma. (Interestingly, d will be comparable to 2^{k-1}.)

We now define the following notion of mutual independence.

Definition 2.2 *For all integers $n > 0$, define $[n] := \{1, \dots, n\}$. Given events $\mathcal{E}_1, \dots, \mathcal{E}_n \subset \Omega$ and a subset $J \subset [n]$, the event $\mathcal{E}_i$ is said to be* mutually independent of $\{\mathcal{E}_j : j \in J\}$ *if for all choices of disjoint subsets $J_1, J_2 \subset J$,*

$$\mathbb{P}\left[\mathcal{E}_i \cap \bigcap_{j_1 \in J_1} \mathcal{E}_{j_1} \cap \bigcap_{j_2 \in J_2} \overline{\mathcal{E}}_{j_2}\right] = \mathbb{P}[\mathcal{E}_i] \cdot \mathbb{P}\left[\bigcap_{j_1 \in J_1} \mathcal{E}_{j_1} \cap \bigcap_{j_2 \in J_2} \overline{\mathcal{E}}_{j_2}\right].$$

Equipped with this notion, we can state the first form of the Lovász Local Lemma, which will help answer the above question of 2-colorability for k-uniform hypergraphs.

Theorem 2.3 (Symmetric Lovász Local Lemma) *Suppose $p \in (0, 1)$, $d \geq 1$, and $\mathcal{E}_1, \dots, \mathcal{E}_n$ are events such that $\mathbb{P}[\mathcal{E}_i] \leq p$ for all i. If each $\mathcal{E}_i$ is mutually independent of all but d other events $\mathcal{E}_j$, and $ep(d + 1) \leq 1$, where $e = 2.71828\dots$ is Euler's number, then $\mathbb{P}\left[\cap_{i=1}^n \overline{\mathcal{E}}_i\right] > 0$.*

Remark 2.4 *In the above result, d is sometimes called the "dependence degree". The "local"-ness of the result has to do with the fact that assumptions depend only on d rather than n, the number of events.*

Before we prove the Local Lemma (in a more general form), let us see how it can be used to study 2-colorability for hypergraphs. In the setting of Example 2.1, suppose the hypergraph G has n edges, denoted by $e_1, \dots, e_n$. Let $\mathcal{E}_i$ denote the event that the edge e_i is monochromatic; as computed above, $p = 2^{1-k}$.

We now claim that "$d = d$", that is, the d in Example 2.1 is precisely the d in Theorem 2.3. Indeed, fix an edge e_i; now any conditioning (i.e., node-coloring) on the edges disjoint from e_i is independent of $\mathcal{E}_i$, since the node colors are i.i.d. Bernoulli random variables. Thus, the assumptions of the Symmetric Lovász Local Lemma are indeed satisfied, as long as

$$d + 1 \leq \frac{1}{ep} = \frac{2^{k-1}}{e}.$$

We stress again that this condition is independent of the number of edges in the hypergraph G.

2.3 (Asymmetric) Lovász Local Lemma: statement and proof

We now prove the Symmetric Lovász Local Lemma, i.e., Theorem 2.3. In fact we show a stronger, "asymmetric" version, and use it to prove the symmetric version. This will require the following useful concept.

Definition 2.5 *A (directed) graph $G = (V(G), E(G))$ is a* dependency (di)graph *on events $\mathcal{E}_1, \dots, \mathcal{E}_n$ if $V(G) = [n]$ and each event $\mathcal{E}_i$ is mutually independent of its non-neighbors $\{\mathcal{E}_j : j \neq i,\ (i, j) \notin E(G)\}$.*

Remark 2.6 *Most applications in the literature use the undirected version of the dependency graph; however, there are some applications that use the digraph structure. In such cases, given a directed edge (i, j), i is the source and j the target.*

We can now state the Lovász Local Lemma in its more general form.

Theorem 2.7 ((Asymmetric) Lovász Local Lemma) *Suppose G is a dependency (di)graph for events $\mathcal{E}_1, \ldots, \mathcal{E}_n$, and there exist $x_1, \ldots, x_n \in (0,1)$ such that*

$$\mathbb{P}[\mathcal{E}_i] \leq x_i \prod_{(i,j)\in E(G)} (1 - x_j), \quad \forall i \in [n]. \tag{2.1}$$

Then,

$$\mathbb{P}\left[\bigcap_{i=1}^{n} \overline{\mathcal{E}}_i\right] \geq \prod_{i=1}^{n}(1 - x_i) > 0. \tag{2.2}$$

Remark 2.8 *Given a set of events $\mathcal{E}_i$, the choice of a dependency digraph G is not unique, nor is the choice of the parameters x_i. Rather, the "user" decides which dependency digraph G and parameters x_i to work with, in a given application. The dependency graph is often clear from the context (e.g. in the hypergraph colorability application above), although the choice of x_i might not be.*

Remark 2.9 *Theorem 2.7 is sharp when the $\mathcal{E}_i$ are independent, G is empty, and $x_i = \mathbb{P}[\mathcal{E}_i]$ $\forall i$.*

Before we show the Asymmetric Local Lemma, let us quickly see why it implies the Symmetric version. Indeed, if the hypotheses of Theorem 2.3 hold, set $x_i = \frac{1}{d+1}$ $\forall i$. Now the hypotheses imply that there is an undirected dependency graph G in which each node has degree at most d. Therefore,

$$\begin{aligned} x_i \prod_{(i,j)\in E(G)} (1 - x_j) = \frac{1}{d+1}\left(1 - \frac{1}{d+1}\right)^{\deg(i)} &\geq \frac{1}{d+1}\left(1 - \frac{1}{d+1}\right)^{d} \\ &\geq \frac{1}{d+1} \cdot \frac{1}{e} \geq \mathbb{P}[\mathcal{E}_i]. \end{aligned}$$

It follows by the Asymmetric Lovász Local Lemma that $\mathbb{P}[\cap_i \overline{\mathcal{E}}_i] > 0$.

Finally, we prove the Asymmetric Lovász Local Lemma.

Proof of Theorem 2.7. Given $S \subset [n]$, define

$$\overline{P}_S := \mathbb{P}\left[\bigcap_{i\in S} \overline{\mathcal{E}}_i\right], \qquad \overline{P}_\emptyset := 1.$$

The result follows once we show, by induction on $|S|$, that for all $S \subset [n]$ and $a \in S$,

$$\frac{\overline{P}_S}{\overline{P}_{S\setminus\{a\}}} \geq 1 - x_a. \tag{2.3}$$

More precisely, we will show by induction on $|S|$ that

$$\overline{P}_S \geq (1 - x_a)\overline{P}_{S\setminus\{a\}} > 0.$$

Indeed, this yields the result, because applying the inequality to $S = [n]$, then $[n-1]$, and so on, yields:

$$\mathbb{P}\left[\bigcap_{i=1}^{n} \overline{\mathcal{E}}_i\right] = \overline{P}_{[n]} \geq (1-x_n)\overline{P}_{[n-1]} = (1-x_n)(1-x_{n-1})\overline{P}_{[n-2]} \geq \cdots \geq \prod_{i=1}^{n}(1-x_i) > 0,$$

as desired.

Thus it remains to prove (2.3). The base case is when $S = \{a\}$ is a singleton. In this case,

$$\frac{\overline{P}_{\{a\}}}{\overline{P}_{\emptyset}} = \mathbb{P}[\overline{\mathcal{E}}_a] \geq 1 - x_a \prod_{(a,j)\in E(G)} (1 - x_j) \geq 1 - x_a,$$

proving the assertion. Now suppose (2.3) holds for all subsets $S' \subset [n]$ with size at most k, and say $S \subset [n]$ has size $k+1$. To proceed further, let us define the neighborhood of $a \in S$, as well as its "closure", via:

$$\Gamma(a) := \{j \in V(G) : (a,j) \in E(G)\}, \qquad \Gamma^+(a) := \{a\} \cup \Gamma(a). \tag{2.4}$$

Now fix $a \in S$, and compute:

$$\overline{P}_S = \mathbb{P}\left[\bigcap_{i\in S} \overline{\mathcal{E}}_i\right] = \mathbb{P}\left[\bigcap_{i\in S\setminus\{a\}} \overline{\mathcal{E}}_i\right] - \mathbb{P}\left[\mathcal{E}_a \cap \bigcap_{i\in S\setminus\{a\}} \overline{\mathcal{E}}_i\right] \geq \mathbb{P}\left[\bigcap_{i\in S\setminus\{a\}} \overline{\mathcal{E}}_i\right] - \mathbb{P}\left[\mathcal{E}_a \cap \bigcap_{i\in S\setminus\Gamma^+(a)} \overline{\mathcal{E}}_i\right]$$
$$= \overline{P}_{S\setminus\{a\}} - \mathbb{P}[\mathcal{E}_a]\overline{P}_{S\setminus\Gamma^+(a)},$$

where the first equality and the inequality are straightforward, and the final equality follows from the mutual independence of $\mathcal{E}_a$ and $\{\mathcal{E}_i : i \notin \Gamma^+(a)\}$. From this computation it follows that

$$\frac{\overline{P}_S}{\overline{P}_{S\setminus\{a\}}} \geq 1 - \mathbb{P}[\mathcal{E}_a]\frac{\overline{P}_{S\setminus\Gamma^+(a)}}{\overline{P}_{S\setminus\{a\}}},$$

where $\overline{P}_{S\setminus\{a\}} > 0$ by the induction hypothesis. Now say $\Gamma(a) \cap S = \{b_1, \dots, b_d\}$ for some $d \geq 0$, and write the fraction on the right-hand side as a telescoping product:

$$\frac{\overline{P}_{S\setminus\Gamma^+(a)}}{\overline{P}_{S\setminus\{a\}}} = \frac{\overline{P}_{S\setminus\{a,b_1\}}}{\overline{P}_{S\setminus\{a\}}}\frac{\overline{P}_{S\setminus\{a,b_1,b_2\}}}{\overline{P}_{S\setminus\{a,b_1\}}} \cdots \frac{\overline{P}_{S\setminus\{a,b_1,\dots,b_d\}}}{\overline{P}_{S\setminus\{a,b_1,\dots,b_{d-1}\}}},$$

where all terms on the right-hand side are strictly positive by the induction hypothesis. By the same hypothesis, each ratio on the right-hand side is bounded above by $\dfrac{1}{1-x_{b_i}}$. Therefore,

$$\frac{\overline{P}_{S\setminus\Gamma^+(a)}}{\overline{P}_{S\setminus\{a\}}} \leq \frac{1}{1-x_{b_1}} \cdots \frac{1}{1-x_{b_d}}.$$

Recalling that by assumption $\mathbb{P}[\mathcal{E}_a] \leq x_a \prod_{b\in\Gamma(a)}(1-x_b)$, it follows that

$$\frac{\overline{P}_S}{\overline{P}_{S\setminus\{a\}}} \geq 1 - x_a \prod_{b\in\Gamma(a)} (1-x_b) \prod_{c\in\Gamma(a)\cap S} \frac{1}{1-x_c} \geq 1 - x_a > 0.$$

This shows (2.3), and with it, the Lovász Local Lemma. □