MATH 233A: Concentration of Measure, Autumn 2018
HOMEWORK 2
Due Monday, November 26

Please try to solve the homework on your own. Discussions are okay but make your own effort.

Problem 1. Suppose that $f : \{0,1\}^n \to \mathbb{R}$ is such that for every $x \in \{0,1\}^n$,

$$\sum_{i=1}^{n} (f(x \oplus e_i) - f(x))^2 \leq v$$

(where $x \oplus e_i$ means flipping the i-th coordinate). Then prove that for X uniform in $\{0,1\}^n$,

$$\mathsf{P}[f(X) \geq \mathbb{E}[f(X)] + t] \leq e^{-2t^2/v}.$$

Hint: $(e^{z/2} - e^{y/2})^2 \leq \frac{1}{8}(z-y)^2(e^z + e^y)$.

Problem 2. Suppose that $X \in \mathbb{R}^n$ has a multivariate Gaussian distribution with covariance matrix Γ. Prove that for any continuously differentiable $f : \mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{Var}[f(X)] \leq \mathbb{E}[(\nabla f(X))^T \Gamma \nabla f(X)],$$

$$\mathrm{Ent}((f(X))^2) \leq 2\mathbb{E}[(\nabla f(X))^T \Gamma \nabla f(X)].$$

Problem 3. Show that non-convex 1-Lipschitz functions (w.r.t. the Euclidean metric) do not satisfy exponential tail bounds with variance $O(1)$ for product distributions on $[0,1]^n$.

Problem 4. Prove that if $f : \mathbb{R}_+ \to \mathbb{R}$ is differentiable and X is exponentially distributed ($\mathsf{P}[X \geq t] = e^{-t}$ for $t \geq 0$), then

$$\mathrm{Ent}((f(X))^2) \leq 4\mathbb{E}[X(f'(X))^2].$$

Hint: Reduce to the Gaussian case by considering $\frac{1}{2}(X_1^2 + X_2^2)$ where X_1, X_2 are independent standard Gaussians.

Problem 5. Let Z denote the number of triangles in a random graph $G_{n,p}$, $p = p(n) \geq 1/n$, $\lim_{n\to\infty} p(n) = 0$. Show that for every $a > 0$ there is $c > 0$ such that for n large enough,

$$\mathsf{P}[Z \geq \mathbb{E}[Z] + an^3p^3] \geq p^{cp^2n^2}.$$

Hint: the right-hand side is the probability that a clique of size $\Theta(pn)$ appears in $G_{n,p}$.

Bonus Problem. Can you derive some form of an exponential tail bound for self-bounding functions from Talagrand's inequality?