Stanford University **Spring 2018**
Math 233A: Non-constructive methods in combinatorics **Instructor: Jan Vondrák**
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Lecture 20. The Kadison-Singer problem, part II

Our goal in this lecture is to prove the following theorem, which will complete the solution of the Kadison-Singer problem.

Theorem 1. *For $A_i \succeq 0$, $Tr\,(A_i) \leq \alpha$, $\sum_{i=1}^m A_i = I$, the maximum root of the (real-rooted) polynomial*

$$\chi\,(x) = \left(1 - \frac{\partial}{\partial z_m}\right) \dots \left(1 - \frac{\partial}{\partial z_1}\right) \left(\det\left(\sum_{i=1}^m z_i A_i\right)\right)\Bigg|_{z_i = x, \forall i}$$

is at most $(1+\sqrt{\alpha})^2$.

Let's first introduce some notation. For a stable polynomial $Q\,(z_1, \dots, z_m)$, the barrier function in direction j is

$$\phi_Q^j\,(z) = \frac{\partial}{\partial z_j}\left[\log\left(Q\,(z)\right)\right].$$

For $i, j \in [n]$, freeze z_k for $k \neq i, j$ and regard $\phi_Q^j\,(z)$ as a polynomial in z_i, for which we denote $\lambda_1, \dots, \lambda_n$ the roots in z_i. Then we can write

$$\phi_Q^j\,(z) = \sum_{l=1}^n \frac{1}{z_j - \lambda_l}.$$

We first show the following lemma. Recall that z being above the roots of Q means that $Q(z+t) > 0$ for any $t \geq 0$. We follow a simplified proof from Terrence Tao's blog.

Lemma 2. *For any stable polynomial Q, if z is above the roots of Q, then $\forall i, j \in [n]$, $\phi_Q^j\,(z)$ is monotone decreasing and convex in z_i.*

Proof. We can assume that Q is a monic polynomial. (The leading coefficient must be positive and hence we can do this by scaling.) Denote $\phi_Q^j\,(x_i, x_j)$ to be the polynomial on $\mathbb{R}^2$ where the other variables $(z_\ell : \ell \notin \{i, j\})$ are frozen in $\mathbb{R}$. We claim that for any positive integer k,

$$(-1)^k \frac{\partial^k}{\partial x_j^k} \phi_Q^i\,(x_i, x_j) \geq 0.$$

To see this, (1) when $i = j$, it is directly from the expression $\phi_Q^i\,(x_i) = \sum_{l=1}^n \frac{1}{x_i - \lambda_l}$. (2) when $i \neq j$, for any fixed x_i, regarding Q as a polynomial in x_j, we have real roots $\lambda_l(x_i)$ for $l \in [n]$. We can write

$$Q(x_i, x_j) = \prod_{l=1}^n \left(x_j - \lambda_l\,(x_i)\right).$$

Since $\phi_Q^i(x_i, x_j) = \frac{\partial}{\partial x_i}\log(Q(x_i, x_j))$, we have that

$$(-1)^k \frac{\partial^k}{\partial x_j^k}\phi_Q^i(x_i, x_j) = (-1)^k \frac{\partial}{\partial x_i}\frac{\partial^k}{\partial x_j^k}\log(Q(x_i, x_j)) = (-1)^{k-1}\frac{1}{(k-1)!}\sum_{l=1}^{n}\frac{1}{(x_j - \lambda_l(x_i))^k}.$$

Note that $\lambda_l(x_i)$ is continuous in x_i. It is known that it is also differentiable as a *complex function* in x_i, except for points of measure 0 (we will accept this claim without proof). We next claim that $\lambda_l(x_i)$ is non-increasing. To see this, if it is increasing, at some point the derivative is positive, let's say at x_i'. Then for a complex z_i close to x_i' with $Im(z_i) > 0$, we have $Im(\lambda_l(z_i)) > 0$, and $Q(z_i, \lambda_l(z_i)) = 0$. It is impossible since $z_i, \lambda_l(z_i)$ are both on the upper half plane, contradicting the fact that Q is stable. Therefore, $\lambda_l(x_i)$ is non-increasing, and

$$(-1)^k \frac{\partial}{\partial x_i}\frac{\partial^k}{\partial x_j^k}\log(Q(x_i, x_j)) \geq 0.$$

□

Next we show the following lemma.

Lemma 3. *If Q is stable and $z \in R^n$ is above its roots, with $\phi_Q^i(z) < 1$, then z is above the roots of $(1 - \frac{\partial}{\partial z_i})Q$.*

Proof. For any $t \geq 0$, we have $\phi_Q^i(t + z) < 1$ by monotonicity. $\phi_Q^i(z) < 1$ implies that $Q'_{z_i}(z)/Q(z) < 1$. Since $Q(z) > 0$, we have that $Q'_{z_i}(z) < Q(z)$, which means that

$$(1 - \frac{\partial}{\partial z_i})Q(z) > 0.$$

□

Lemma 4. *If Q is stable, z is above its roots, with $\phi_Q^i(z) \leq 1 - \frac{1}{\delta}$ for $\delta > 1$. Then*

$$\phi^i_{(1-\frac{\partial}{\partial z_j})Q}(z + \delta e_j) \leq \phi_Q^i(z).$$

Proof. We have that

$$\phi^i_{(1-\frac{\partial}{\partial z_j})Q} = \frac{\frac{\partial}{\partial z_i}(Q - Q'_{z_j})}{Q - Q'_{z_j}} = \frac{\partial_i Q}{Q} + \frac{\partial_i\left(1 - \phi_Q^j\right)}{1 - \phi_Q^j} = \phi_Q^i - \frac{\partial_j \phi_Q^i}{1 - \phi_Q^j}, \tag{1}$$

where we write ∂_i for $\frac{\partial}{\partial z_i}$ for short. Note that

$$\partial_j \phi_Q^i(z + \delta) \geq \partial_j \phi_Q^i(z_j) \geq \frac{\phi_Q^i(z_j + \delta) - \phi_Q^i(z_j)}{\delta}.$$

By (1) we have that

$$\phi^i_{(1-\frac{\partial}{\partial z_j})Q}(z_j + \delta) = \phi_Q^i(z_j + \delta) - \frac{\partial_j \phi_Q^i(z_j + \delta)}{1 - \phi_Q^j(z_j + \delta)}. \tag{2}$$

By our condition we have that

$$1-\phi_Q^j\left(z_j+\delta\right)\geq 1-\phi_Q^j\left(z_j\right)\geq \frac{1}{\delta},$$

and thus

$$\frac{\partial_j\phi_Q^i\left(z_j+\delta\right)}{1-\phi_Q^j\left(z_j+\delta\right)}\geq \delta\partial_j\phi_Q^i\left(z_j+\delta\right)\geq \phi_Q^i\left(z_j+\delta\right)-\phi_Q^i\left(z_j\right). \tag{3}$$

Substituting (3) into (2) we see that

$$\phi^i_{(1-\frac{\partial}{\partial z_j})Q}\left(z_j+\delta\right)\leq \phi_Q^i\left(z_j\right).$$

□

Now we are ready to prove Theorem 1.

Proof of Theorem 1 Let

$$Q_k(z_1,\ldots,z_m)=\left(1-\frac{\partial}{\partial z_k}\right)...\left(1-\frac{\partial}{\partial z_1}\right)\left[\det\left(\sum_{i=1}^m z_iA_i\right)\right].$$

We claim that barrier functions are bounded by $1-\frac{1}{\delta}$ for $\delta=1+\sqrt{\alpha}$, if with each $\left(1-\frac{\partial}{\partial z_k}\right)$ operation, we increase t in the respective coordinate by δ. We use induction to show this. In the base case we have that

$$Q_0\left(z_1,\ldots,z_m\right)=\det\left(\sum_{i=1}^m z_iA_i\right),$$

and thus

$$Q_0\left(t,\ldots,t\right)=\det\left(\sum_{i=1}^m tA_i\right)=t^n>0.$$

Recall that

$$\frac{d}{dt}\left(\det\left(A+tB\right)\right)|_{t=0}=\det\left(A\right)Tr\left(A^{-1}B\right).$$

Using the above identity we see that

$$\phi^i_{Q_0}\left(z_1,\ldots,z_m\right)=Tr\left(\left(\sum_{i=1}^m z_iA_i\right)^{-1}A_i\right),$$

which implies that

$$\phi^i_{Q_0}\left(t,\ldots,t\right)=\frac{1}{t}Tr\left(A_i\right)\leq \alpha/t,$$

since we assumed that $Tr\left(A_i\right)<\alpha$. Letting $t=\alpha+\sqrt{\alpha}$, we further see that

$$\phi^i_{Q_0}\left(t,\ldots,t\right)\leq 1-\frac{1}{1+\sqrt{\alpha}}=1-\frac{1}{\delta}.$$

Now assume that the claim holds for k. We have that

$$\begin{aligned}
& \phi^i_{Q_{k+1}}\left(\underbrace{t+\delta,\ldots,t+\delta}_{k+1},t,\ldots,t\right) \\
= \; & \phi^i_{(1-\partial_{k+1})Q_k}\left(\underbrace{t+\delta,\ldots,t+\delta}_{k+1},t,\ldots,t\right) \\
\leq \; & \phi^i_{Q_k}\left(\underbrace{t+\delta,\ldots,t+\delta}_{k},t,\ldots,t\right) \\
\leq \; & 1-\frac{1}{\delta}.
\end{aligned}$$

Therefore we see that $\chi(x) = Q_m(x,\ldots,x)$ has maximum root at most $t+\delta = (1+\sqrt{\alpha})^2$.