MATH 233B: Polyhedral techniques in combinatorial optimization
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1 Extended Formulations

Suppose $P \subset \mathbb{R}^n$ is some polytope, and suppose it's defined by exponentially many constraints (aka facets). Maybe we can express P as a projection of $Q \subset \mathbb{R}^{n+n'}$ in such a way that Q has only polynomially many constraints and variables. An example is the permutahedron, defined as follows:

$$P_{perm}^{(n)} = \text{conv}(\{(\pi(1), \pi(2), \ldots, \pi(n) : \pi \in S_n\}) \subset \mathbb{R}^n$$

Equivalently, in terms of constraints,

$$\begin{aligned} P_{perm}^{(n)} &= \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i = \binom{n+1}{2}; \forall S \subseteq [n], \sum_{i \in S} x_i \geq \binom{|S|+1}{2} \right\} \\ &= \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i = \binom{n+1}{2}; \forall T \subseteq [n], \sum_{i \in T} x_i \leq \binom{n+1}{2} - \binom{n-|T|+1}{2} \right\} \end{aligned}$$

With this last formulation, we see that much like matroid polytopes, the permutahedron is defined in terms of a submodular function. That's because, as we saw, a function that depends only on the cardinality of a set is submodular if and only if that function is concave, which is indeed the case for the right-hand side in the latter description. This is an example of a so-called *polymatroid* polytope. We can still do the uncrossing trick for tight constraints, which implies that for any vertex, there is a chain of n tight inequalities. In particular, this implies that any vertex is indeed the vector of a permutation, which implies that this is a valid constraint representation of the permutahedron. All of these inequalities are needed, which means we have exponentially many constraints. However, we can save by increasing the dimension.

Consider the permutation matrix Y_π whose entries are

$$y_{ij} = \begin{cases} 1 & j = \pi(i) \\ 0 & \text{otherwise} \end{cases}$$

Then $\text{conv}(\{Y_\pi\})$ is just the bipartite matching polytope for $K_{n,n}$. That has a description in terms of constraints as

$$y_{ij} \geq 0; \forall i, \sum_{j=1}^n y_{ij} = 1; \forall j, \sum_{i=1}^n y_{ij} = 1$$

We can also reintroduce the x variables as

$$x_i = \sum_{j=1}^n j \cdot y_{ij}$$

In other words,

$$P_{perm}^{(n)} = \left\{ x : \exists y \geq 0 \in \mathbb{R}^{n \times n}, \forall i, x_i = \sum_{j=1}^{n} j \cdot y_{ij}; \forall i, \sum_{j=1}^{n} y_{ij} = 1; \forall j, \sum_{i=1}^{n} y_{ij} = 1 \right\}$$

So we increased the dimension from n to $n + n^2$, and the number of constraints went from 2^n to $n^2 + 3n$. In this case, our upper polytope Q would consist of all pairs $(x, y) \in \mathbb{R}^{n+n^2}$ that satisfy all these constraints.

Definition 1 *The extension complexity of P is the minimum number of inequalities in an extended formulation Q of P.*

Can we always do this? Probably not, just by computational considerations: there are NP-hard problems that can be encoded as polytopes, and if we could lift them like this then we'd get a poly-time algorithm for them. For instance, the cut polytope of a graph is defined by

$$P_{cut}(G) = \mathrm{conv}(\{\delta_G(S) : S \subseteq V(G)\})$$

where $\delta_G(S) = \{(u, v) \in E : |S \cap \{u, v\}| = 1\}$. Then if we could maximize a linear function over P_{cut}, then we could solve the max-cut problem, which is NP-hard. The key point here is that if a polytope has polynomial extension complexity, then we can solve LPs over it in polynomial time, since optimizing over Q gives us an algorithm for optimizing over P. Then the natural question is to prove that the extension complexity of P_{cut} (say) is super-polynomial or exponential. This research program was initiated by Yannakakis in 1991, and has seen renewed interest in the past few years. The first breakthrough was by Fiorini, Massar, Pokutta, Tiwary, and de Wolf in 2011, who proved an exponential lower bound for P_{cut} and other related polytopes, including the correlation polytope and the TSP polytope. The new intuition came from quantum communication complexity.

Before the lower bounds, we'll see a nice positive result. Recall that the spanning tree polytope of G is given by

$$P_{sp-tree}(G) = \{x \in \mathbb{R}^E : x \geq 0; x(E[V]) = |V| - 1; \forall W \subseteq V, x(E[W]) \leq |W| - 1\}$$

Theorem 2

$$P_{sp-tree}(G) = \left\{ x \in \mathbb{R}^E : \begin{array}{l} \exists z_{v,w,u}, (v, w, u) \text{ an ordered triple of distinct vertices with } \{v, w\} \in E; \\ \forall \{v, w\} \in E, \forall u \notin \{v, w\}, x_{\{v,w\}} = z_{v,w,u} + z_{w,v,u}; \\ x_{\{v,w\}} + \sum_{u \in V \setminus \{v,w\}, \{v,\} \in E} z_{v,u,w} = 1; \\ x_{\{v,w\}} \geq 0, z_{v,w,u} \geq 0 \end{array} \right\}$$

The interpretation of $z_{v,w,u}$ is that "u is on the side of w in the spanning tree."

Proof: Call the right-hand polytope P'. If T is a spanning tree (and thus χ_T is a vertex of $P_{sp-tree}(G)$), then $x_{\{v,w\}} = 1$ if $\{v, w\} \in T$, and 0 otherwise. Then $z_{v,w,u} = 1$ if w is on the unique path between v and u in T, and $\{v, w\} \in T$, and 0 otherwise. We need to check that all constraints are satisfied. If $x_{\{v,w\}} = 1$, then exactly one of $z_{v,w,u}, z_{w,v,u}$ is 1, since u must be on one of the two sides of the edge $\{v, w\}$ in T. So the first constraint is satisfied. Additionally, if $x_{\{v,w\}} = 1$, then all $z_{v,u,w} = 0$, since there can be no vertex between v, w in T. Finally, if $x_{\{v,w\}} = 0$ then there is exactly one u such that $\{v, u\} \in T$ and u is on the path $v \to w$, namely the first vertex along that path. That means that $z_{v,u,w} = 1$ for that u and 0 for all others. Thus, the second constraint is always satisfied. Therefore, $P_{sp-tree}(G) \subseteq P'$.

Conversely, let $x \in P'$ and z the corresponding vector of additional variables. We want to prove that x satisfies the constraints defining $P_{sp-tree}$. For simplicity, we will only do it for $G = K_n$. So pick $W \subseteq V$, and assume $|W| \geq 3$. Then

$$
\begin{aligned}
x(E[W]) &= \sum_{\substack{v<w \\ v,w \in W}} x_{\{v,w\}} \\
&= \frac{1}{|W|-2} \sum_{\substack{v<w \\ v,w \in W}} \sum_{u \in W \setminus \{v,w\}} (z_{v,w,u} + z_{w,v,u}) \\
&= \frac{1}{|W|-2} \sum_{u,v,w \in W \text{ distinct}} z_{v,w,u} \\
&= \frac{1}{|W|-2} \sum_{\substack{u,v \in W \\ u \neq v}} \sum_{w \in W \setminus \{u,v\}} z_{v,w,u} \\
&\leq \frac{1}{|W|-2} \sum_{\substack{u,v \in W \\ u \neq v}} (1 - x_{\{u,v\}}) \\
&= \frac{1}{|W|-2} \left(|W|(|W|-1) - 2 \sum_{\{u,v\} \in E[W]} x_{\{u,v\}} \right)
\end{aligned}
$$

Rearranging, we get that

$$\frac{|W|}{|W|-2} x(E[W]) \leq \frac{|W|(|W|-1)}{|W|-2}$$

which is what we wanted. Moreover, the only inequality is actually an equality when $W = V$, so we indeed get $x \in P_{sp-tree}(G)$. □

Because of this, one might hope that every problem in P has small extension complexity. However, Rothvoss proved in 2014 that the non-bipartite matching polytope has exponential extension complexity.

There are various LP hierarchies, which are ways of taking an LP, increasing the dimension, producing some new inequalities, and hopefully getting a more useful LP. The two main LP hierarchies are Lovász-Schrijver and Sherali-Adams; additionally, the Laserre hierarchy (aka sum-of-squares) does the same thing for SDP problems.