MATH 233B: Polyhedral techniques in combinatorial optimization
Instructor: Jan Vondrák
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1 Totally Unimodular Matrices

Definition 1 (Totally Unimodular Matrix) *A matrix* $\mathbf{A}$ *is totally unimodular if every square submatrix has determinant* 0, $+1$, *or* -1. *In particular, this implies that all entries are* 0 *or* ± 1.

Totally unimodular matrices are very well behaved, because they always define polytopes with integer vertices, as long as the right-hand side is integer-valued.

Theorem 2 *If* $\mathbf{A}$ *is totally unimodular and* $\mathbf{b}$ *is an integer vector, then* $P = \{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ *has integer vertices.*

Proof: Let $\mathbf{v}$ be a vertex of P. As we discussed, there exists a non-singular square submatrix $\mathbf{A}'$ of $\mathbf{A}$ such that $\mathbf{A}'\mathbf{v} = \mathbf{b}'$. We have $\det \mathbf{A}' = \pm 1$ since $\mathbf{A}'$ is nonsingular. By Cramer's Rule, we have $v_i = \frac{\det(\mathbf{A}'_i|\mathbf{b})}{\det \mathbf{A}'}$ where $\mathbf{A}'_i|\mathbf{b}$ is $\mathbf{A}'$ with the i-th column replaced by $\mathbf{b}$. Therefore, v_i is an integer. □

Lemma 3 *For all bipartite graphs* G, *the incidence matrix* $\mathbf{A}$ *is totally unimodular.*

Proof: Recall that $\mathbf{A}$ is a 0-1 matrix, where columns are indexed by edges and each column has exactly two 1's, corresponding to the two vertices of the edge. We proceed by induction. The claim is certainly true for a 1×1 matrix.

Assume the claim holds true for all $(k-1) \times (k-1)$ submatrices. Let $\mathbf{A}'$ be a $k \times k$ submatrix of $\mathbf{A}$. Each column in $\mathbf{A}'$ has at most two 1's. If any column has no 1's, it must have all 0's, and the matrix is singular. If any column has exactly one nonzero entry, then $\det \mathbf{A}' = \pm \det \mathbf{A}''$, where $\mathbf{A}''$ is obtained by deleting the respective row and column; we have $\det \mathbf{A}'' \in \{0, \pm 1\}$ by induction.

$$\begin{array}{c|c|c} \ddots & 0 & \iddots \\ \hline \cdots & 1 & \cdots \\ \hline \iddots & 0 & \ddots \end{array}$$

Otherwise, every column has exactly two 1's. In particular, since G is bipartite, the rows can be partitioned into V_1, V_2 such that for each column, there is exactly one 1 in V_1 and in V_2. Then by summing up all the rows corresponding to V_1 and subtracting the rows corresponding to V_2, we get $\mathbf{0}$. Therefore, $\mathbf{A}'$ is singular and $\det \mathbf{A}' = 0$.

$$\begin{array}{r} V_1 \left\{ \begin{array}{c|c|c} \ddots & \vdots & \iddots \\ \cdots & 1 & \cdots \\ \iddots & \vdots & \ddots \end{array} \right. \\ \hline V_2 \left\{ \begin{array}{c|c|c} \ddots & \vdots & \iddots \\ \cdots & 1 & \cdots \\ \iddots & \vdots & \ddots \end{array} \right. \end{array} \quad \begin{array}{c} + \\ + \\ + \\ - \\ - \\ - \end{array}$$

□

Lemma 4 *If* $\mathbf{A}$ *is totally unimodular, then* $\left[\frac{\mathbf{A}}{\mathbf{I}}\right]$ *is totally unimodular.*

Proof: By the determinant expansion formula, the determinant of any square submatrix $\mathbf{A}'$ is equal to 0 or $\pm \det \mathbf{A}''$ where $\mathbf{A}''$ is a square submatrix of $\mathbf{A}$ (see the figure). By definition, $\det \mathbf{A}'' \in \{0, \pm 1\}$.

$$\left[\begin{array}{ccc|c} & \cdots & & \mathbf{A}'' \\ \hline 1 & & & \\ & \ddots & & \vdots \\ & & 1 & \end{array}\right]$$

□

2 Generalized König's Theorem

Definition 5 (b-matching) *A* $\mathbf{b}$*-matching is an assignment* $\mathbf{x} : E \to \mathbb{Z}_+$ *such that for all* $v \in V, x(\delta(v)) \leq b_v$.

Definition 6 (c-vertex cover) *A* $\mathbf{c}$*-vertex cover is an assignment* $\mathbf{y} : V \to \mathbb{Z}_+$ *such that for all edges* $e = (u, v), y_u + y_v \geq c_e$.

Let $\mathbf{b} \in \mathbb{Z}_+^V$ and $\mathbf{c} \in \mathbb{Z}_+^E$. We have

$$\max\{\mathbf{c}^T\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} = \min\{\mathbf{b}^T\mathbf{y} : \mathbf{A}^T\mathbf{y} \geq \mathbf{c}, \mathbf{y} \geq \mathbf{0}\}.$$

By the total unimodularity of $\mathbf{A}$, vertices of these polyhedra are integer vectors, so optimal solutions can be taken as integer. (The polyhedron in the dual is unbounded, but the optimum does not change if we constrain the polyhedron for example by $y_v \leq \max c_e$ for all $v \in V$. Then we get a bounded integer polytope and hence there is an integer optimum.) The primal can be interpreted as a maximum $\mathbf{b}$-matching and the dual as a minimum $\mathbf{c}$-vertex cover.

Theorem 7 (Generalized König's Theorem) *For all bipartite graphs with* $\mathbf{b} \in \mathbb{Z}_+^V, \mathbf{c} \in \mathbb{Z}_+^E$, *the Max* $\mathbf{c}$*-weighted* $\mathbf{b}$*-matching is equal to the Min* $\mathbf{b}$*-weighted* $\mathbf{c}$*-vertex covers.*

3 Maximum flow

Another important class of problems for which the relevant matrices are totally unimodular are *flow problems*. While we are not going to cover flows in the detail that they deserve, we want to mention how total unimodularity plays a role here.

Definition 8 *For a directed graph* G *and a vertex* v, *we denote* $\delta_{out}(v) = \{(v, w) : (v, w) \in E\}$ *and* $\delta_{in}(v) = \{(u, v) : (u, v) \in E\}$. *Similarly, for a set of vertices* U, $\delta_{out}(U) = \{(u, v) \in E : u \in U, v \notin V\}$ *and* $\delta_{in}(W) = \{(v, w) \in E : v \notin W, w \in W\}$.

Definition 9 *For a directed graph G with two special vertices s, t, and edge capacities c_e, an s-t flow is an assignment $x : E \to \mathbb{R}$ such that*

- *For each edge e, $0 \leq x_e \leq c_e$.*
- *For each vertex $v \neq s, t$, $\sum_{e \in \delta_{in}(v)} x_e = \sum_{e \in \delta_{out}}(v)$.*

The Maximum Flow problem is the problem of finding an s-t flow maximizing $\sum_{e \in \delta_{out}(v)} x_e$.

We can write down this problem in matrix notation as follows. Let A be the signed incidence matrix of G, where $A_{v,e} = 1$, $A_{u,e} = -1$ for $e = (u, v)$, and $A_{w,e} = 0$ for $w \notin e$.

Lemma 10 *The signed adjacency matrix of a directed graph is totally unimodular.*

(Note that there is no assumption of bipartiteness here.)

Proof: Exactly the same as for bipartite graphs. In the last case, when every column of A' contains two nonzero entries, we observe that the rows sum to 0, so $det(A') = 0$. □

Let us formulate the Max Flow problem in matrix notation. Let A' denote A with the rows corresponding to s, t removed. The flow conservation condition can be written as $A'\mathbf{x} = 0$. Let $\mathbf{w}$ denote the row corresponding to t. Then we get the following LP:

$$\max\{\mathbf{w}^T \mathbf{x} : 0 \leq \mathbf{x} \leq \mathbf{c}, A'\mathbf{x} = 0\}.$$

From the total unimodularity of A, we obtain the following.

Corollary 11 *For $\mathbf{c} \in \mathbb{Z}^E$, there is an optimal flow with integer values.*

We can also easily derive the classical Max-flow Min-cut Theorem.

Definition 12 *An s-t cut is any set of edges C such that there is no directed s-t path in $E \setminus C$. The capacity of C is $\sum_{e \in C} c_e$.*

Theorem 13 *The maximum flow of an s-t cut is equal to the minimum capacity of an s-t cut.*

Proof: From LP duality, we get

$$\max\{\mathbf{w}^T \mathbf{x} : A'\mathbf{x} = 0, 0 \leq \mathbf{x} \leq \mathbf{c}\} = \min\{\mathbf{c}^T \mathbf{y} : A'^T \mathbf{z} + \mathbf{y} \geq \mathbf{w}, \;\; \mathbf{z} \in \mathbb{R}^{V \setminus \{s,t\}}, \mathbf{y} \in \mathbb{R}^E_+\}.$$

Since A' is TUM, we get integral optimal solutions $\mathbf{x}^*$ and $(\mathbf{y}^*, \mathbf{z}^*)$ for the primal and dual LP. Let us simplify the description a little bit by extending $\mathbf{z}^*$ to a vector in $\mathbb{R}^V$, where $z^*_s = 0$ and $z^*_t = -1$. We get $A^T \mathbf{z}^* + \mathbf{y}^* = A'^T \mathbf{z}^* - \mathbf{w} + \mathbf{y}^* \geq 0$. Observe that $\mathbf{y}^*$ should be as small as possible, which considering this constraint for $e = (u, v)$ means $y^*_{uv} = \max\{z^*_u - z^*_v, 0\}$.

Define $U = \{u \in V : z^*_u \geq 0\}$. We have $s \in U, t \notin U$, so $\delta_{out}(U)$ is an s-t cut. Since z^*_u are integers, we have $y^*_{uv} \geq z^*_u - z^*_v \geq 1$ for each $(u, v) \in \delta_{out}(U)$. Hence, $OPT = \mathbf{c}^T \mathbf{y} \geq \sum_{e \in \delta_{out}(U)} c_e$. Clearly, $\sum_{e \in \delta_{out}(U)} c_e \geq \sum_{e \in \delta_{out}(s)} x_e = OPT$, so all the inequalities are equalities. □