Optimization of Submodular Functions
Tutorial - lecture I

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Lecture I: outline

1. Submodular functions: what and why?
2. Convex aspects: Submodular minimization
3. Concave aspects: Submodular maximization

Combinatorial optimization

There are many problems that we study in combinatorial optimization...
Max Matching, Min Cut, Max Cut, Min Spanning Tree, Max SAT, Max Clique, Vertex Cover, Set Cover, Max Coverage, ...

They are all problems in the form

$$\max\{f(S) : S \in \mathcal{F}\}$$

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We can

- try to deal with each problem individually, or
- try to capture some properties of $f, \mathcal{F}$ that make it tractable.

Continuous optimization

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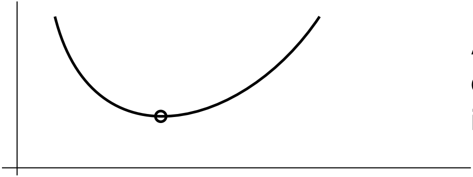

A function $f : \mathbb{R}^n \to \mathbb{R}$
can be minimized efficiently,
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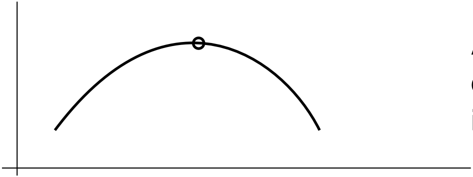

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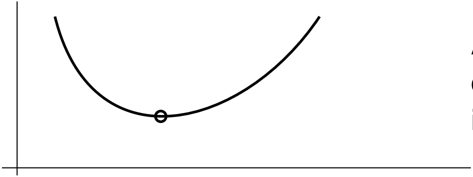

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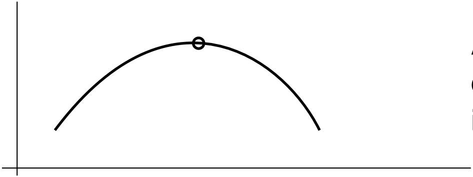

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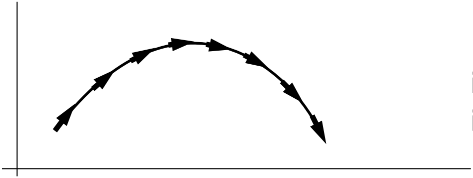
Discrete analogy?
Not so obvious... f is now a set function, or equivalently

$$f : \{0, 1\}^n \to \mathbb{R}.$$

From concavity to submodularity

Concavity:



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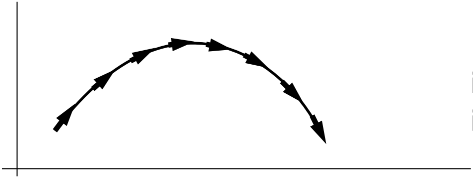

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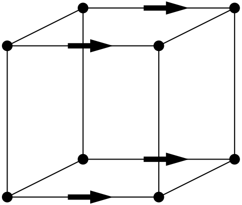
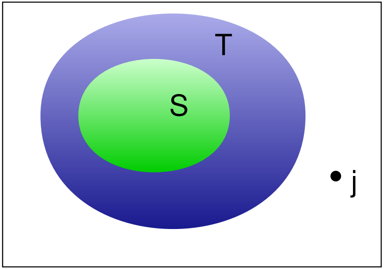


$f : \{0,1\}^n \to \mathbb{R}$ is submodular,

if $\forall i$, the discrete derivative $\partial_i f(x) = f(x + e_i) - f(x)$ is non-increasing in x.

(1) Define the *marginal value of element j*,

$$f_S(j) = f(S \cup \{j\}) - f(S).$$



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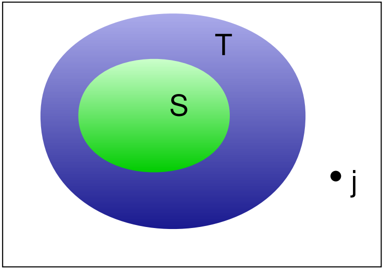

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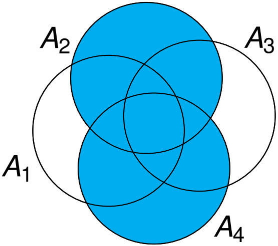
(2) A function $f : 2^N \to \mathbb{R}$ is submodular if for any S, T,

$$f(S \cup T) + f(S \cap T) \leq f(S) + f(T).$$

Examples of submodular functions

Coverage function:
Given $A_1, \ldots, A_n \subset U$,

$$f(S) = \left| \bigcup_{j \in S} A_j \right|.$$



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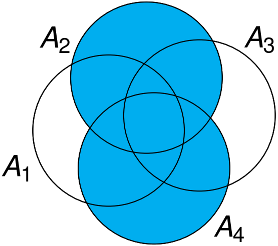

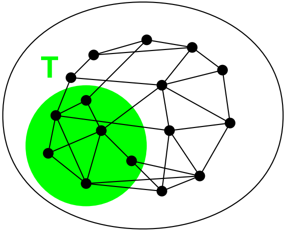

Cut function:

$$\delta(T) = |e(T, \overline{T})|$$

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- **Argument for concavity:** Definition looks more like concavity - *non-increasing* discrete derivatives.
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Theorem (Grötschel-Lovász-Schrijver, 1981; Iwata-Fleischer-Fujishige / Schrijver, 2000)

There is an algorithm that computes the minimum of any submodular function $f : \{0,1\}^n \to \mathbb{R}$ *in* $poly(n)$ *time (using value queries,* $f(S) = ?$*).*

In contrast:
Maximizing a submodular function (e.g. Max Cut) is NP-hard.

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Convex aspects of submodular functions

Why is it possible to minimize submodular functions?

- The combinatorial algorithms are sophisticated...
- But there is a simple explanation: the *Lovász extension*.

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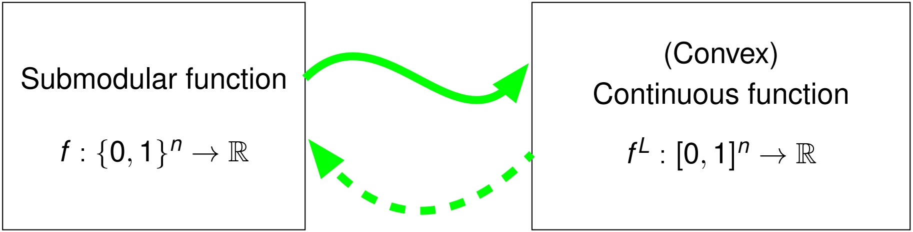

- If f is submodular, then f^L is convex.
- Therefore, f^L can be minimized efficiently.
- A minimizer of $f^L(x)$ can be converted into a minimizer of $f(S)$.

The Lovász extension

Definition

Given $f : \{0,1\}^n \to \mathbb{R}$, its Lovász extension $f^L : [0,1]^n \to \mathbb{R}$ is

$$f^L(x) = \sum_{i=0}^{n} \alpha_i f(S_i)$$

where $x = \sum \alpha_i \mathbf{1}_{S_i}, \sum \alpha_i = 1, \alpha_i \geq 0$ and $\emptyset = S_0 \subset S_1 \subset \ldots \subset S_n$.

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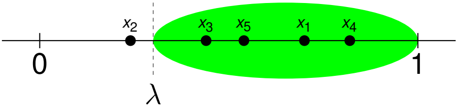
Equivalently:



$f^L(x) = \mathbb{E}[f(T_\lambda(x))]$,
where $T_\lambda(x) = \{i : x_i > \lambda\}$,
$\lambda \in [0,1]$ uniformly random.

Minimizing the Lovász extension

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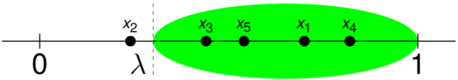

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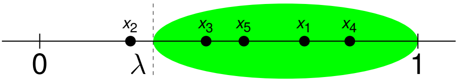

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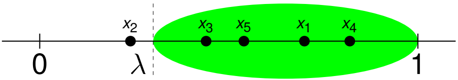

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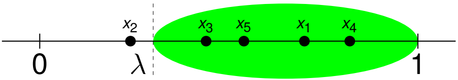

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- f is submodular $\Leftrightarrow$ f^L is convex (in fact the "convex closure" of f).
- Therefore, f^L can be minimized (by the ellipsoid method, in weakly polynomial time).
- Given a minimizer of $f^L(x)$, we get a convex combination $f^L(x) = \sum_{i=0}^{n} \alpha_i f(T_i)$, and one of the T_i is a minimizer of $f(S)$.

Generalized submodular minimization

Submodular functions can be minimized over restricted families of sets:

- lattices, odd/even sets, T-odd sets, T-even sets [Grötschel, Lovász, Schrijver '81-'84]
- "parity families", including $\mathcal{L}_1 \setminus \mathcal{L}_2$ for lattices $\mathcal{L}_1, \mathcal{L}_2$ [Goemans, Ramakrishnan '95]
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However, a simple "covering" constraint can make submodular minimization hard:

- $\min\{f(S) : |S| \geq k\}$
- $\min\{f(T) : T \text{ is a spanning tree in } G\}$
- $\min\{f(P) : P \text{ is a shortest path between } s - t\}$

What about approximate solutions?

Constrained submodular minimization

Bad news:

$\min\{f(S) : S \in \mathcal{F}\}$ becomes very hard for some simple constraints:

- $n^{1/2}$-hardness for $\min\{f(S) : |S| \geq k\}$ [Goemans,Harvey,Iwata,Mirrokni '09], [Svitkina,Fleischer '09]
- $n^{2/3}$-hardness for $\min\{f(P) : P \text{ is a shortest path}\}$ [Goel,Karande,Tripathi,Wang '09]
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Good news:

sometimes $\min\{f(S) : S \in \mathcal{F}\}$ is equally hard for linear/submodular f:

- Variants of Facility Location [Svitkina,Tardos '06], [Chudak,Nagano '07]
- 2-approximation for $\min\{f(S) : S \text{ is a vertex cover}\}$ [Koufagiannis,Young; Iwata,Nagano; GKTW '09]
- 2-approximation for Submodular Multiway Partition (generalizing Node-weighted Multiway Cut) [Chekuri,Ene '11]

Submodular Vertex Cover

Submodular Vertex Cover: $\min\{f(S) : S \subseteq V \text{ hits every edge in } G\}$
- formulation using the Lovász extension:

$$\begin{aligned} \min f^L(x) &: \\ \forall (i,j) \in E; x_i + x_j &\geq 1; \\ x &\geq 0. \end{aligned}$$

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  S is a vertex cover because each edge has a variable $x_i \geq 1/2$.
- Expected cost of the solution is

$$\mathbb{E}[f(S)] = 2 \int_0^{1/2} f(T_\lambda(x)) d\lambda \leq 2 \int_0^1 f(T_\lambda(x)) d\lambda = 2f^L(x).$$

Submodular Multiway Partition

Submodular Multiway Partition: $\min \sum_{i=1}^{k} f(S_i)$ where $(S_1, \ldots, S_k)$ is a partition of V, and $i \in S_i$ for $i \in \{1, 2, \ldots, k\}$ (k terminals).

$$\min \sum_{i=1}^{k} f^L(x_i) :$$

$$\forall j \in V; \sum_{i=1}^{k} x_{ij} = 1;$$

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$(2 - 2/k)$*-approximation algorithm:*

- Given a fractional solution x, let $A_i = T_\lambda(x_i)$, where $\lambda \in [\frac{1}{2}, 1]$ is uniformly random. Let $U = V \setminus \bigcup_{i=1}^{k} A_i$ be the unallocated vertices.
- Return $S_{i'} = A_{i'} \cup U$ for a random i', and $S_i = A_i$ for $i \neq i'$.

Submodular minimization overview

Constraint	Approximation	Hardness	alg. technique
Unconstrained	1	1	combinatorial
Parity families	1	1	combinatorial
Vertex cover	2	2	Lovász ext.
k-unif. hitting set	k	k	Lovász ext.
Multiway k-partition	$2 - 2/k$	$2 - 2/k$	Lovász ext.
Facility location	$\log n$	$\log n$	combinatorial
Set cover	n	$n/\log^2 n$	trivial
$\|S\| \geq k$	$\tilde{O}(\sqrt{n})$	$\tilde{\Omega}(\sqrt{n})$	combinatorial
Shortest path	$O(n^{2/3})$	$\Omega(n^{2/3})$	combinatorial
Spanning tree	$O(n)$	$\Omega(n)$	combinatorial

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Submodular maximization

Maximization of submodular functions:

- comes up naturally in allocation / welfare maximization settings
- $f(S) =$ value of a set of items S ... often submodular due to combinatorial structure or property of *diminishing returns*
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Hence, we distinguish:

1. Monotone submodular maximization:
   e.g. $\max\{f(S) : |S| \leq k\}$, generalizing Max k-cover.
2. Non-monotone submodular maximization:
   e.g. $\max f(S)$, generalizing Max Cut.

Monotone submodular maximization

Theorem (Nemhauser,Wolsey,Fisher '78)

The greedy algorithm gives a $(1 - 1/e)$*-approximation for the problem* $\max\{f(S) : |S| \leq k\}$ *where* f *is monotone submodular.*

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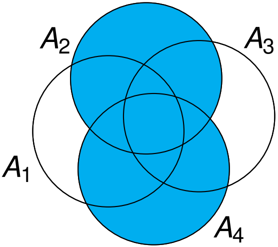
Generalizes a greedy $(1 - 1/e)$-approximation for Max k-cover:

<u>Max k-cover</u>

Choose k sets so as to maximize $\left|\bigcup_{j \in K} A_j\right|$.



[Feige '98]:
Unless $P = NP$*, there is no* $(1 - \frac{1}{e} + \epsilon)$*-approximation for Max* k*-cover.*

Analysis of Greedy

Greedy Algorithm: S_i = *solution after* i *steps;*
pick next element a *to maximize* $f(S_i + a) - f(S_i)$.

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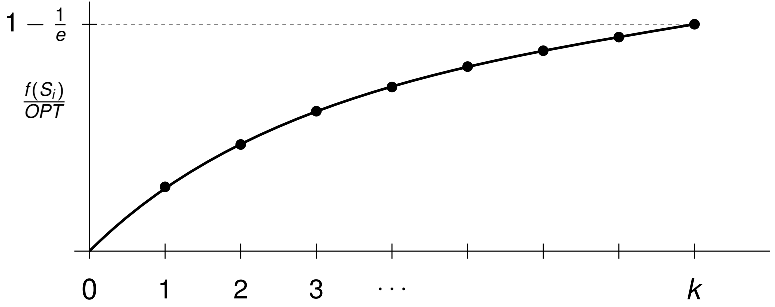
Greedy Algorithm: *S_i = solution after i steps; pick next element a to maximize $f(S_i + a) - f(S_i)$.*

Let the optimal solution be S^*. By submodularity:

$$\exists a \in S^* \setminus S_i;\ f(S_i + a) - f(S_i) \geq \frac{1}{k}(OPT - f(S_i)).$$



$$OPT - f(S_{i+1}) \leq (1 - \tfrac{1}{k})(OPT - f(S_i))$$

$$\Rightarrow OPT - f(S_k)) \leq (1 - \tfrac{1}{k})^k OPT \leq \tfrac{1}{e} OPT.$$

Submodular maximization under a matroid constraint

Nemhauser, Wolsey and Fisher considered a more general problem:

Given: Monotone submodular function f, matroid $\mathcal{M} = (N, \mathcal{I})$.
Goal: Find $S \in \mathcal{I}$ maximizing $f(S)$.

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The greedy algorithm gives a $\frac{1}{2}$-approximation for the problem $\max\{f(S) : S \in \mathcal{I}\}$.

More generally: $\frac{1}{k+1}$-approximation for the problem $\max\{f(S) : S \in \mathcal{I}_1 \cap \mathcal{I}_2 \cap \ldots \cap \mathcal{I}_k\}$.

Motivation: what are matroids and what can be modeled using a matroid constraint?

Matroids

Definition

A matroid on N is a system of *independent sets* $\mathcal{I} \subset 2^N$, satisfying

1. $\forall B \in \mathcal{I}, A \subset B \Rightarrow A \in \mathcal{I}$.
2. $\forall A, B \in \mathcal{I}, |A| < |B| \Rightarrow \exists x \in B \setminus A; \ A \cup \{x\} \in \mathcal{I}$.

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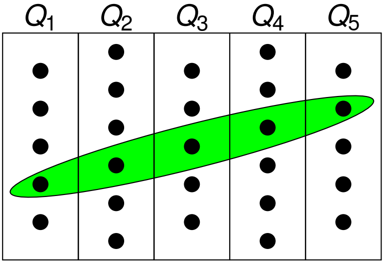

Example: *partition matroid*

S is independent, if $|S \cap Q_i| \leq 1$ for each Q_i.

Submodular Welfare $\rightarrow$ matroid constraint

Submodular Welfare Maximization:
Given n players with submodular valuation functions $w_i : 2^M \rightarrow \mathbb{R}_+$. *Partition* $M = S_1 \cup \ldots \cup S_n$ *so as to maximize* $\sum_{i=1}^{n} w_i(S_i)$.

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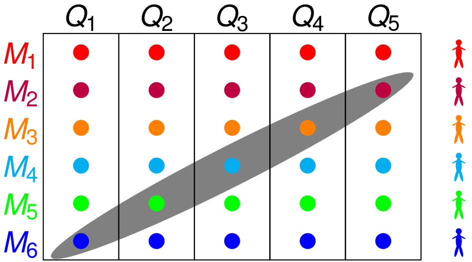

Reduction:

Create n clones of each item,
$f(S) = \sum_i w_i(S \cap M_i)$,
$\mathcal{I} = \{S : \forall i; |S \cap Q_i| \leq 1\}$
(a partition matroid).

Submodular Welfare Maximization is equivalent to $\max\{f(S) : S \in \mathcal{I}\}$
$\Rightarrow$ Greedy gives $\frac{1}{2}$-approximation.

Further combinatorial techniques

Partial enumeration: "guess" the first t elements, then run greedy.

- $(1 - 1/e)$-approximation for monotone submodular maximization subject to a knapsack constraint, $\sum_{j \in S} w_j \leq B$ [Sviridenko '04]

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Local search: switch up to t elements, as long as it provides a (non-trivial) improvement; possibly iterate in several phases.

- $1/3$-approximation for unconstrained (non-monotone) maximization [Feige,Mirrokni,V. '07]
- $1/(k + 2 + \frac{1}{k} + \delta_t)$-approximation for non-monotone maximization subject to k matroids [Lee,Mirrokni,Nagarajan,Sviridenko '09]
- $1/(k + \delta_t)$-approximation for *monotone* submodular maximization subject to $k \geq 2$ matroids [Lee,Sviridenko,V. '10]

Continuous relaxation for submodular maximization?

Questions that don't seem to be answered by combinatorial algorithms:

- What is the optimal approximation for $\max\{f(S) : S \in \mathcal{I}\}$, in particular the *Submodular Welfare Problem*?
- What is the optimal approximation for *multiple constraints*, e.g. multiple knapsack constraints?
- In general, how can we combine *different types of constraints*?

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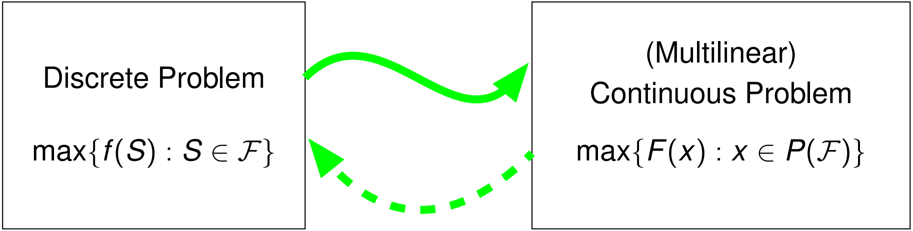
It would be nice to have a *continuous relaxation*, but:

1. The *Lovász extension* is convex, therefore not suitable for maximization.
2. The counterpart of the convex closure is the *concave closure*

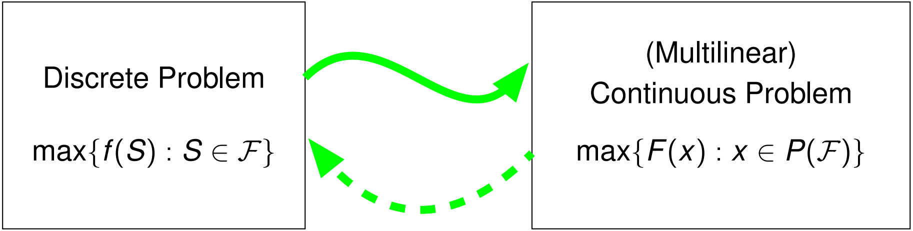
$$f^+(x) = \max\{\sum \alpha_S f(S) : \sum \alpha_S \mathbf{1}_S = x, \sum \alpha_S = 1, \alpha_S \geq 0\}.$$

However, this extension is NP-hard to evaluate!

Multilinear relaxation



Multilinear relaxation



Multilinear extension of f:

- $F(x) = \mathbb{E}[f(\hat{x})]$, where $\hat{x}$ is obtained by rounding each x_i randomly to $0/1$ with probabilities x_i.
- $F(x)$ is neither convex nor concave; it is multilinear and $\frac{\partial^2 F}{\partial {x_i}^2} = 0$.
- $F(x + \lambda \vec{d})$ is a *concave* function of λ, if $\vec{d} \geq 0$.

Algorithms based on the multilinear relaxation

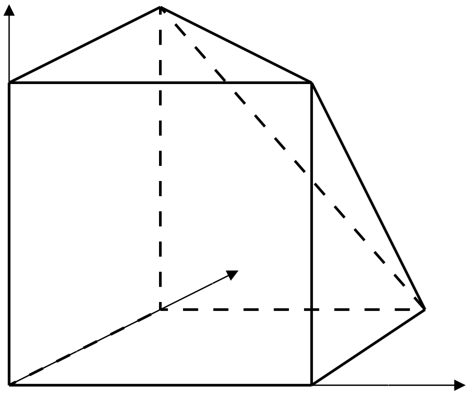
The multilinear relaxation turns out to be useful for maximization:

1. **The continuous problem** $\max\{F(x) : x \in P\}$ can be solved:
   - $(1 - 1/e)$-approximately for any monotone submodular function and solvable polytope [V. '08]
   - $(1/e)$-approximately for any nonnegative submodular function and downward-closed solvable polytope [Feldman,Naor,Schwartz '11]

   (earlier constant factors: 0.325 [Chekuri,V.,Zenklusen '11], 0.13 [Fadaei,Fazli,Safari '11])
2. **A fractional solution can be rounded:**
   - without loss for a matroid constraint [Calinescu,Chekuri,Pál,V. '07]
   - losing $(1 - \epsilon)$ factor for a constant number of knapsack constraints [Kulik,Shachnai,Tamir '10]
   - losing $O(k)$ factor for k matroid constraints, in a modular fashion (to be combined with other constraints) [Chekuri,V.,Zenklusen '11]
   - e.g., $O(k)$-approximation for k matroids & $O(1)$ knapsacks

The Continuous Greedy Algorithm

Problem: $\max\{F(x) : x \in P)\}$.



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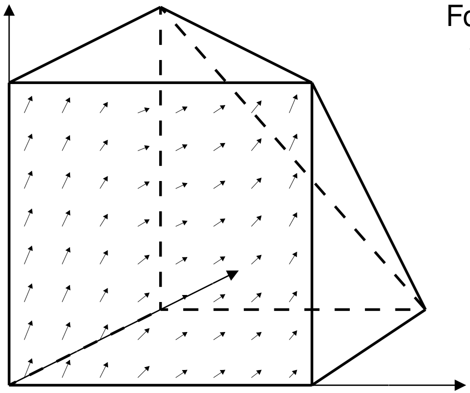

For each $x \in P$, define $v(x)$ by
$v(x) = argmax_{v \in P}(v \cdot \nabla F|_x)$.

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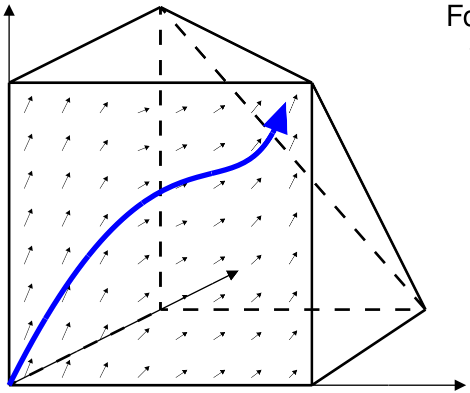

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$$x(0) = 0$$

$$\frac{dx}{dt} = v(x)$$

Run this process
for $t \in [0, 1]$ and return $x(1)$.

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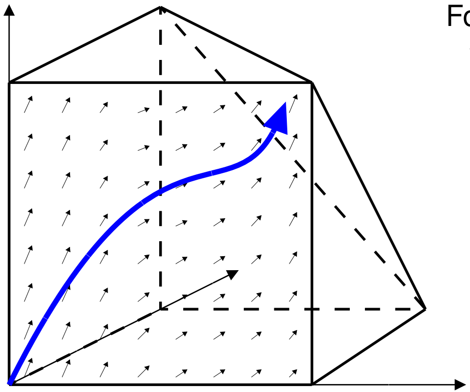

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Claim: $x(1) \in P$ and $F(x(1)) \geq (1 - 1/e)OPT$.

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Evolution of the fractional solution:

- Differential equation: $x(0) = 0$, $\frac{dx}{dt} = v(x)$.

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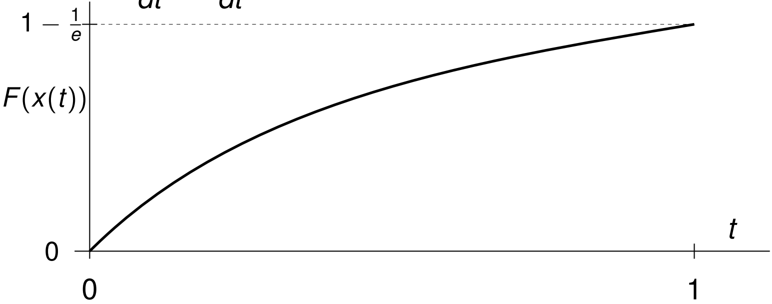

Solve the differential equation:

$$F(x(t)) \geq (1 - e^{-t}) \cdot OPT.$$

Submodular maximization overview

MONOTONE MAXIMIZATION

Constraint	Approximation	Hardness	technique
$\|S\| \leq k$	$1 - 1/e$	$1 - 1/e$	greedy
matroid	$1 - 1/e$	$1 - 1/e$	multilinear ext.
$O(1)$ knapsacks	$1 - 1/e$	$1 - 1/e$	multilinear ext.
k matroids	$k + \epsilon$	$k/\log k$	local search
k matroids & $O(1)$ knapsacks	$O(k)$	$k/\log k$	multilinear ext.

NON-MONOTONE MAXIMIZATION

Constraint	Approximation	Hardness	technique
Unconstrained	$1/2$	$1/2$	combinatorial
matroid	$1/e$	0.48	multilinear ext.
$O(1)$ knapsacks	$1/e$	0.49	multilinear ext.
k matroids	$k + O(1)$	$k/\log k$	local search
k matroids & $O(1)$ knapsacks	$O(k)$	$k/\log k$	multilinear ext.