Rectangles Are Nonnegative Juntas

(An approach to communication lower bounds)

Mika Göös, Shachar Lovett, Raghu Meka,
Thomas Watson, and David Zuckerman

Alice

$x \in \{0,1\}^n$

Bob

$y \in \{0,1\}^n$

Alice

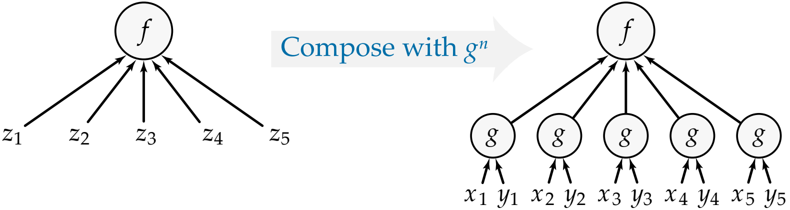
$x \in \{0,1\}^n$

Bob

$y \in \{0,1\}^n$

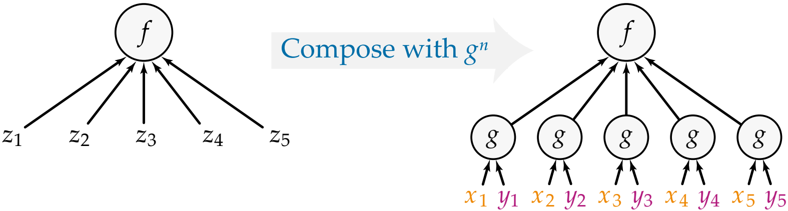
Compute: $F(x,y)$

Composed functions $f \circ g^n$



Examples:
- Set-disjointness: $\mathrm{OR} \circ \mathrm{AND}^n$
- Inner-product: $\mathrm{XOR} \circ \mathrm{AND}^n$
- Equality: $\mathrm{AND} \circ \neg\mathrm{XOR}^n$

Composed functions $f \circ g^n$

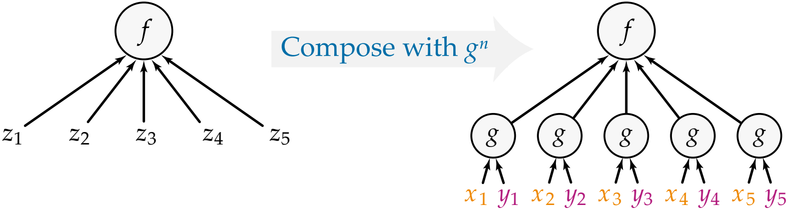


In general: $g\colon \{0,1\}^b \times \{0,1\}^b \to \{0,1\}$ is a small gadget

- **Alice** holds $x \in (\{0,1\}^b)^n$
- **Bob** holds $y \in (\{0,1\}^b)^n$

Inputs x and y encode $z := g^n(x,y)$

Composed functions $f \circ g^n$



Holy grail (Conjecture):

Simulate **cost-*d*** randomised protocol for $f \circ g^n$
using **height-*d*** randomised decision tree for f

i.e., $$\mathbf{BPP}^{\mathbf{cc}}(f \circ g^n) \approx \mathbf{BPP}^{\mathbf{dt}}(f)$$

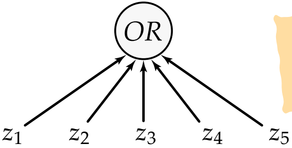
Composed functions $f \circ g^n$

Holy grail (Conjecture):

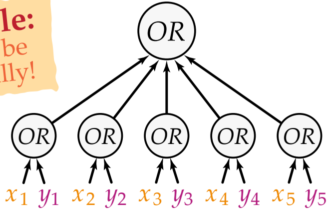
Simulate **cost-*d*** randomised protocol for $f \circ g^n$
using **height-*d*** randomised decision tree for f

i.e., $$\mathbf{BPP}^{\mathbf{cc}}(f \circ g^n) \approx \mathbf{BPP}^{\mathbf{dt}}(f)$$

Composed functions $f \circ g^n$



Bad example:
Gadget must be chosen carefully!



Our result:

Simulate **cost-*d*** randomised protocol for $f \circ g^n$
using ~~height-d randomised decision tree~~ for f

... **degree-*d* conical junta** ...

Main structure theorem

Conical *d*-junta:

Nonnegative combination of d-conjunctions

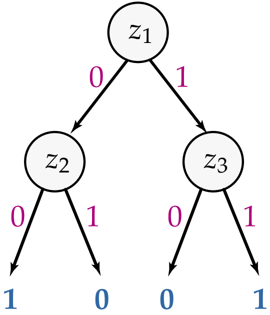
EXAMPLE: $0.4 \cdot z_1\bar{z}_2 + 0.66 \cdot z_2\bar{z}_3 + 0.35 \cdot z_3\bar{z}_1$

Main structure theorem

Conical d-junta:

Nonnegative combination of d-conjunctions

EXAMPLE: $0.4 \cdot z_1\bar{z}_2 + 0.66 \cdot z_2\bar{z}_3 + 0.35 \cdot z_3\bar{z}_1$



$\rightsquigarrow \quad \bar{z}_1\bar{z}_2 + z_1z_3$

Main structure theorem

Conical d-junta:

Nonnegative combination of d-conjunctions

EXAMPLE: $0.4 \cdot z_1\bar{z}_2 + 0.66 \cdot z_2\bar{z}_3 + 0.35 \cdot z_3\bar{z}_1$

Junta Theorem:

- (f is *any* partial function)
- g is inner-product on $\Theta(\log n)$ bits
- Π is cost-d randomised protocol for $f \circ g^n$

$$\Downarrow$$

There exists a conical d-junta h s.t. $\forall z \in \{0,1\}^n$:

$$\Pr_{(x,y)\sim(g^n)^{-1}(z)}[\,\Pi(x,y) \text{ accepts}\,] \approx h(z)$$

Main structure theorem

Conical d-junta:

Nonnegative combination of d-conjunctions

EXAMPLE: $0.4 \cdot z_1\bar{z}_2 + 0.66 \cdot z_2\bar{z}_3 + 0.35 \cdot z_3\bar{z}_1$

Junta Theorem:

- (f is *any* partial function)
- g is inner-product on $\Theta(\log n)$ bits
- Π is cost-d randomised protocol for $f \circ g^n$

$$\Downarrow$$

There exists a conical d-junta h s.t. $\forall z \in \{0,1\}^n$:

$$\Pr_{(x,y)\sim(g^n)^{-1}(z)}[\,\Pi(x,y) \text{ accepts}\,] \;\approx\; h(z)$$

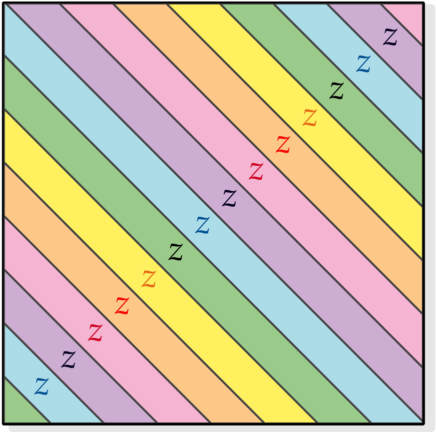
Cf:
- **Polynomial approximation** [Razborov, Sherstov, Shi–Zhu,...]
- **Sherali–Adams vs. LPs** [Chan–Lee–Raghavendra–Steurer]

Junta Theorem in pictures

0 1 1 1 1 1 0 1 1 0 1 0 1 0 0 0 1 1 0 1 1 0 0 1 1 0 1 1 0
0 1 0 0 1 0 0 1 0 1 0 1 0 1 1 1 0 1 0 1 0 0 0 1 0 0 1 1 1
1 1 0 1 1 1 1 0 1 0 1 0 0 1 1 0 1 0 1 0 1 1 0 0 1 1 0 0 0
0 1 1 1 1 0 1 0 1 1 1 1 0 1 0 1 0 1 0 1 0 1 0 0 0 1 0 0 1
0 0 0 1 1 1 0 1 1 0 0 1 1 0 1 1 1 0 1 0 1 1 0 0 0 0 0 1 0
0 1 0 1 0 1 1 1 1 0 0 0 0 1 0 0 0 0 0 0 1 1 1 0 0 1 0 1 1
0 0 0 1 1 0 0 1 0 0 1 1 0 1 1 0 1 0 1 1 0 1 0 0 0 1 1 1 0
1 1 0 1 0 1 1 1 1 1 0 0 0 1 1 0 0 0 1 0 0 1 0 1 1 0 0 1 0
0 1 0 1 0 0 0 0 1 0 1 1 0 0 1 0 0 1 1 0 1 1 1 0 1 0 0 1 1
0 0 0 1 0 0 1 1 1 0 0 1 1 1 0 0 1 0 1 1 0 0 1 1 0 0 0 0 1
0 1 0 1 0 1 0 0 1 1 0 1 0 0 1 1 1 1 1 1 1 1 0 0 0 1 1 0 0
1 1 1 1 1 0 1 1 0 1 0 1 0 1 0 1 1 0 1 0 0 0 1 0 0 0 1 1 0
0 0 1 0 1 1 1 1 0 0 1 0 0 0 1 1 1 1 1 1 1 0 1 1 1 0 1 1 0
1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 0 1 1 0 1 0 1 1 0 0 0 0 1 1
1 1 0 1 0 1 1 0 0 0 1 1 0 1 0 0 1 1 0 1 0 1 1 1 1 0 1 1 1
0 0 1 1 1 0 1 0 0 1 0 0 1 1 1 1 1 0 0 0 0 1 0 0 0 0 1 1 0
0 1 0 0 1 0 1 0 1 1 0 1 1 1 1 0 1 0 0 1 0 1 0 1 1 0 0 1 1
1 1 0 0 0 0 0 1 1 1 0 1 1 0 0 0 0 1 1 0 1 0 1 0 1 1 1 1 1
0 0 1 1 1 0 1 0 0 1 0 0 1 0 0 0 1 0 0 0 0 0 1 1 1 0 1 0 0
1 0 0 1 0 0 0 1 0 0 0 0 0 1 1 0 1 0 0 0 0 0 1 0 1 1 0 0 0
1 0 1 1 0 1 1 0 1 0 0 1 0 0 1 0 1 1 0 0 1 0 1 0 0 0 0 1 0

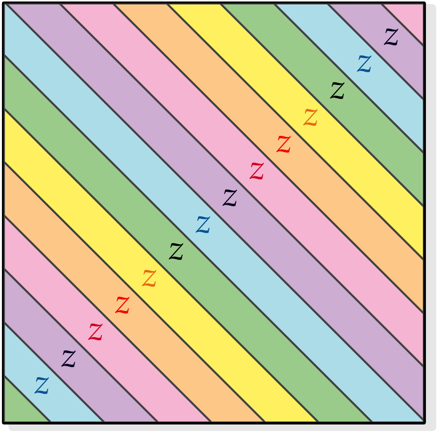
Communication matrix of $f \circ g^n$

Junta Theorem in pictures



Communication matrix of g^n

Junta Theorem in pictures

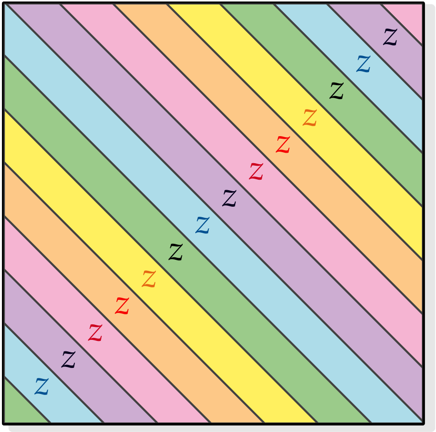


Communication matrix of g^n

Encode $z \in \{0,1\}^n$ randomly:

$$(x, y) \sim (g^n)^{-1}(z)$$

Junta Theorem in pictures



Communication matrix of g^n

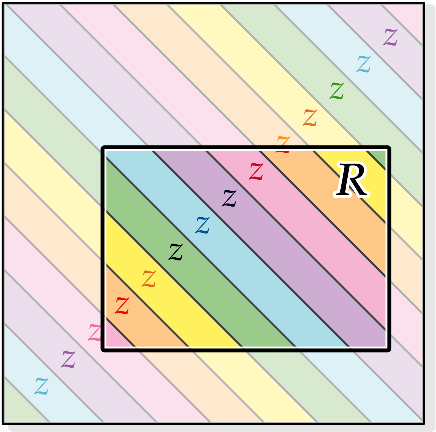
Encode $z \in \{0,1\}^n$ randomly:

$$(x, y) \sim (g^n)^{-1}(z)$$

Want to understand

$$\Pr[\,\Pi(x, y) \text{ accepts}\,]$$

Junta Theorem in pictures



Communication matrix of g^n

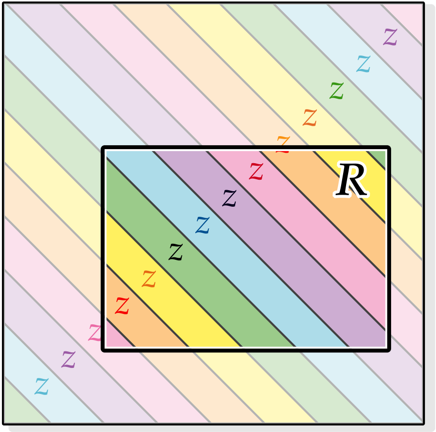
Encode $z \in \{0,1\}^n$ randomly:

$$(x, y) \sim (g^n)^{-1}(z)$$

Want to understand

$$\Pr[\,(x, y) \in R\,]$$

Junta Theorem in pictures



Communication matrix of g^n

Encode $z \in \{0,1\}^n$ randomly:

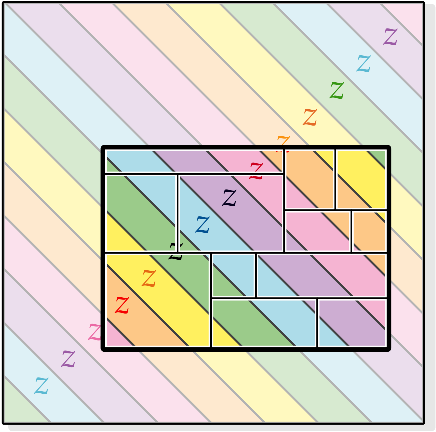
$$(x, y) \sim (g^n)^{-1}(z)$$

Main Theorem:

$\exists$ conical d-junta h,

$$\Pr[\,(x, y) \in R\,] \approx h(z)$$

Junta Theorem in pictures



Communication matrix of g^n

Encode $z \in \{0,1\}^n$ randomly:

$$(x, y) \sim (g^n)^{-1}(z)$$

Main Theorem:

$\exists$ conical d-junta h,

$$\Pr[\,(x, y) \in R\,] \approx h(z)$$

Proof: Partition R into "conjunctions" R':

$$g^n(R') = 1\,1\,0 * * * * * *$$

Corollaries—Simulation Theorems

Communication-to-query simulation for NP:

$$\mathbf{NP^{cc}}(f \circ g^n) = \mathbf{NP^{dt}}(f) \cdot \Theta(b)$$

…recall $b = \Theta(\log n)$

Corollaries—Simulation Theorems

Communication-to-query simulation for NP:

$$\mathbf{NP}^{\mathbf{cc}}(f \circ g^n) = \mathbf{NP}^{\mathbf{dt}}(f) \cdot \Theta(b)$$

…recall $b = \Theta(\log n)$

Conical d-junta: $0.4 \cdot z_1\bar{z}_2 + 0.66 \cdot z_2\bar{z}_3 + 0.35 \cdot z_3\bar{z}_1$

⇝

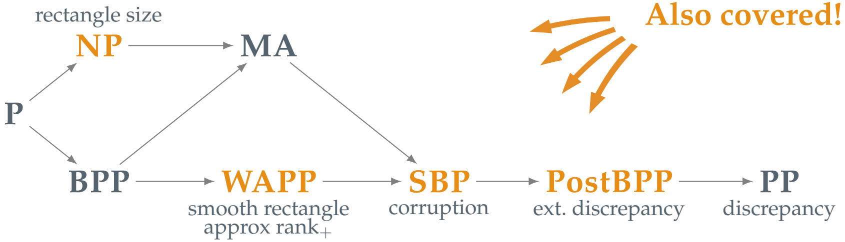
d-DNF: $z_1\bar{z}_2 \vee z_2\bar{z}_3 \vee z_3\bar{z}_1$

Corollaries—Simulation Theorems

Communication-to-query simulation for NP:

$$\mathbf{NP}^{\mathbf{cc}}(f \circ g^n) = \mathbf{NP}^{\mathbf{dt}}(f) \cdot \Theta(b)$$

…recall $b = \Theta(\log n)$

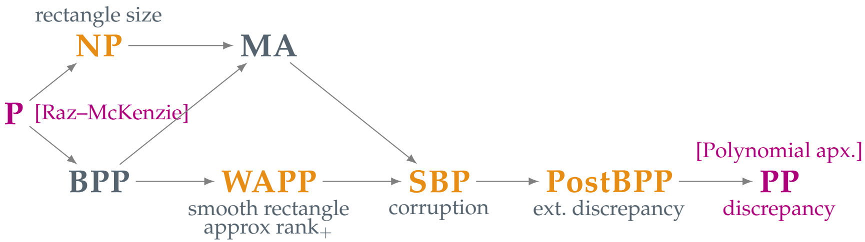


Corollaries—Simulation Theorems

Communication-to-query simulation for NP:

$$\mathbf{NP}^{\mathbf{cc}}(f \circ g^n) = \mathbf{NP}^{\mathbf{dt}}(f) \cdot \Theta(b)$$

…recall $b = \Theta(\log n)$



Resolving open problems

Query lower bound $\rightsquigarrow$ Communication lower bound

1. From [Böhler–Glaßer–Meister'06]:
   $\mathbf{SBP}^{\mathbf{cc}}$ is not closed under intersection

SBP: Small bounded-error computations

- ***yes***-inputs accepted with prob. $\geq \alpha$
- ***no***-inputs accepted with prob. $\leq \alpha/2$

Resolving open problems

Query lower bound $\rightsquigarrow$ Communication lower bound

1. From [Böhler–Glaßer–Meister'06]:
   $\mathbf{SBP^{cc}}$ is not closed under intersection

2. From [Klauck'03]:
   Corruption does not characterise $\mathbf{MA^{cc}}$
   i.e., $\mathbf{MA^{cc}} \subsetneq \mathbf{SBP^{cc}}$

Resolving open problems

Query lower bound $\rightsquigarrow$ Communication lower bound

1. From [Böhler–Glaßer–Meister'06]:
   $\mathbf{SBP}^{\mathbf{cc}}$ is not closed under intersection

2. From [Klauck'03]:
   Corruption does not characterise $\mathbf{MA}^{\mathbf{cc}}$
   i.e., $\mathbf{MA}^{\mathbf{cc}} \subsetneq \mathbf{SBP}^{\mathbf{cc}}$

3. From [Kol–Moran–Shpilka–Yehudayoff'14]:
   No efficient error amplification for $\epsilon\text{-rank}_+$

4. From [Yannakakis'88]: ***(Subsequent work)***
   Clique vs. Independent Set problem
   i.e., $\mathbf{coNP}^{\mathbf{cc}}(F) \gg \mathbf{UP}^{\mathbf{cc}}(F)$

Summary

Main result: **Junta Theorem**

- Let g = inner-product on $b = \Theta(\log n)$ bits
- Let $(\boldsymbol{x}, \boldsymbol{y}) \sim (g^n)^{-1}(z)$

$\Longrightarrow$ $\Pr[\,\Pi(\boldsymbol{x}, \boldsymbol{y}) \text{ accepts}\,] \approx$ Conical junta of z

Open problems

- More applications of **Junta Theorem**?
- Simulation theorems for **BPP**?
- Improve gadget size down to $b = O(1)$
  (Would give new proof of $\Omega(n)$ bound for set-disjointness)

Summary

Main result: **Junta Theorem**

- Let $g =$ inner-product on $b = \Theta(\log n)$ bits
- Let $(\boldsymbol{x}, \boldsymbol{y}) \sim (g^n)^{-1}(z)$

$\Longrightarrow$ $\Pr[\,\Pi(\boldsymbol{x}, \boldsymbol{y})\text{ accepts}\,] \approx$ Conical junta of z

Open problems

- More applications of **Junta Theorem**?
- Simulation theorems for **BPP**?
- Improve gadget size down to $b = O(1)$
  (Would give new proof of $\Omega(n)$ bound for set-disjointness)

Cheers!