Roots of Unity

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1 Introduction/facts you should know

1. (Roots of unity) Let $n \geq 2$ be an integer and let $\zeta = e^{2\pi i/n} = \cos(2\pi/n) + i\sin(2\pi/n)$. Then $1, \zeta, \zeta^2, \ldots, \zeta^{n-1}$ are called the *nth roots of unity*.
2. If ζ is an nth root of unity, ζ satisfies $\zeta^n - 1 = 0$. If $\zeta \neq 1$, then $1+\zeta+\zeta^2+\cdots+\zeta^{n-1} = 0$.
3. Let $\zeta = e^{2\pi i/n}$. Then $x^n - 1 = (x-1)(x-\zeta)(x-\zeta^2)\cdots(x-\zeta^{n-1})$.
4. Let $\zeta = e^{2\pi i/n}$. Then $1 + x + x^2 + \cdots + x^{n-1} = (x-\zeta)(x-\zeta^2)\cdots(x-\zeta^{n-1})$.
5. $1 + \zeta^m + \zeta^{2m} + \cdots + \zeta^{(n-1)m}$ is n if $n|m$, and 0 otherwise.
6. (Primitive root of unity) ζ is a *primitive nth root of unity* if it is an nth root of unity and $1, \zeta, \ldots, \zeta^{n-1}$ are all distinct.
7. (Primitive root of unity v2) $\zeta = e^{2\pi ik/n}$ is a primitive nth root of unity iff $\gcd(k, n) = 1$.
8. (Cyclotomic polynomial) The nth cyclotomic polynomial, $\Phi_n(x)$, is the polynomial whose roots are the nth primitive roots of unity.
9. (Cyclotomic polynomial facts) The nth cyclotomic polynomial is the unique irreducible polynomial with integer coefficients that is a divisor of $x^n - 1$ and not a divisor of $x^k - 1$ for any $k < n$.

   The nth cyclotomic polynomial is the *minimal polynomial* for the nth primitive roots of unity, i.e. for each primitive nth root ζ, $\Phi_n(x)$, the monic polynomial with integer coefficients of minimum degree with ζ as a root.
10. (Geometry) The roots of unity form the vertices of a regular n-gon on the unit circle in the complex plane. Multiplying complex numbers by $\zeta = e^{2\pi i/n}$ corresponds to rotations of $2\pi/n$ about the origin.
11. (Trigonometry) If $\zeta = e^{i\theta}$, then $\cos(\theta) = \frac{\zeta+\zeta^{-1}}{2}$ and $\sin(\theta) = \frac{\zeta-\zeta^{-1}}{2i}$.
12. (Roots of unity filter) Let $f(x) = \sum_i a_i x^i$ be a polynomial and $\zeta = e^{2\pi i/n}$. Then

$$\frac{f(x) + f(\zeta x) + f(\zeta^2 x) + \cdots + f(\zeta^{n-1}x)}{n} = \sum_{n|i} a_i x^i. \tag{1}$$

2 Problem for discussion

1. Let n be an odd positive integer, and let $\zeta = \cos(2\pi/n) + i\sin(2\pi/n)$. Evaluate

$$\frac{1}{1+1} + \frac{1}{1+\zeta} + \frac{1}{1+\zeta^2} + \cdots + \frac{1}{1+\zeta^{n-1}}.$$

2. Verify (1).

3. Evaluate

$$\binom{2017}{0} + \binom{2017}{3} + \cdots + \binom{2017}{2016}.$$

4. (USAMO 1976) If $P(x), Q(x), R(x)$ are polynomials such that $P(x^5)+xQ(x^5)+x^2R(x^5)$ is divisible by $1+x+x^2+x^3+x^4$, prove that $P(x)$ is divisible by $x-1$.

5. Let a, b, c, d be positive integers, such that an $a \times b$ rectangle can be tiled with a combination of $c \times 1$ vertical strips, and $1 \times d$ horizontal strips. Prove that either a is divisible by c or b is divisible by d.

6. Show that if a rectangle can be tiled by smaller rectangles each of which has at least one integer side, then the tiled rectangle has at least one integer side.

7. (IMO 1995) Let p be an odd prime. Find the number of p-element subsets of $\{1, \ldots, 2p\}$ with sum divisible by p.

8. (China 2010) Let $M = \{1, 2, \cdots, n\}$, each element of M is colored in either red, blue or yellow. Set

   $A = \{(x, y, z) \in M \times M \times M | x + y + z \equiv 0 \mod n,\ x, y, z \text{ are of same color}\}$,

   $B = \{(x, y, z) \in M \times M \times M | x + y + z \equiv 0 \mod n,\ x, y, z \text{ are of pairwise distinct color}\}$.

   Prove that $2|A| \geq |B|$.

3 Additional practice

9. (USAMO 1996) Prove that the average of the numbers $n \sin n^\circ$ for $n = 2, 4, 6, \ldots, 180$ is $\cot 1^\circ$.

10. Prove that $\cos(\pi/7)$ is a root of a cubic polynomial with integer coefficients.

11. (Putnam) Let $P_0 = (0, 0)$. For $n = 1, 2, \cdots, 8$, let P_n be the rotation of point P_{n-1} 45 degrees counter-clockwise around the point $(n, 0)$. Determine the coordinates of P_8.

12. (USAMO 1977) Find all pairs of positive integers (m, n) such that the polynomial $1 + x^n + x^{2n} + \cdots + x^{mn}$ is divisible by the polynomial $1 + x + x^2 + \cdots + x^m$.

13. (Leningrad Mathematical Olympiad 1991) A sequence $a_1, a_2, ..., a_n$ is called k-balanced if $a_1 + a_{k+1} + \cdots = a_2 + a_{k+2} + \cdots = \cdots = a_k + a_{2k} + \cdots$. Suppose the sequence $a_1, a_2, \ldots, a_{50}$ is k-balanced for $k = 3, 5, 7, 11, 13, 17$. Prove that all the values a_i are zero.

14. (India 2015) Let $\omega = e^{2\pi i/5}$. Show that there are no positive integers $a_1, a_2, a_3, a_4, a_5, a_6$ such that the following is an integer:

$$(1 + a_1\omega)(1 + a_2\omega)(1 + a_3\omega)(1 + a_4\omega)(1 + a_5\omega)(1 + a_6\omega).$$

15. The set of integers is partitioned into a finite number of arithmetic progressions with each integer in one progression. Prove that some two of these progressions have the same common difference.

16. (Jim Propp) n light bulbs are located at the vertices of a regular n-gon. Initially, exactly one bulb is turned on. You can choose any collection of bulbs whose positions are the vertices of a regular polygon, and if they are all on, you can turn them all off, or if they are all off, you can turn them all on. For what n can you eventually get all of the bulbs on?

17. Let a_k, b_k, c_k be integers for $k = 1 \ldots n$, and let $f(x)$ be the number of ordered triples (A, B, C) of subsets (possibly empty) of the set $S = \{1, \ldots, n\}$ that partition S for which
$$\sum_{i \in S \setminus A} a_i + \sum_{i \in S \setminus B} b_i + \sum_{i \in S \setminus C} c_i \equiv x \pmod 3.$$
Suppose that $f(0) = f(1) = f(2)$. Prove that there exists $i \in S$ such that $3|a_i + b_i + c_i$.

18. (IMO Shortlist 2007) Find all positive integers n such that you can color the numbers in the set $S = \{1, \ldots, n\}$ red or blue such that the set $S \times S \times S$ contains exactly 2007 monochromatic ordered triples (x, y, z) for which $n|x + y + z$.

19. If P is a nonzero polynomial with integer coefficients, show that there exists a complex number z with $|z| = 1$ and $|P(z)| \geq 1$.

20. Let $P(x)$ be a monic polynomial with integer coefficients such that all its zeros lie on the unit circle. Show that all the zeros of $P(x)$ are roots of unity, i.e., $P(x)|(x^n - 1)^k$ for some $n, k \in N$.

21. Let

$$A(x) = \sum_{k \geq 0} \frac{x^{3k}}{(3k)!}, \qquad B(x) = \sum_{k \geq 0} \frac{x^{3k+1}}{(3k+1)!}, \qquad C(x) = \sum_{k \geq 0} \frac{x^{3k+2}}{(3k+2)!}.$$

Evaluate $A(x)^3 + B(x)^3 + C(x)^3 - 3A(x)B(x)C(x)$.