CS364A: Problem Set #3

Due to the TAs by noon on Friday, November 8, 2013

Instructions:

(0) We'll grade this assignment out of a total of 75 points; if you earn more than 75 points on it, the extra points will be treated as extra credit.

(1) Form a group of at most 3 students and solve as many of the following problems as you can. You should turn in only one write-up for your entire group.

(2) Turn in your solutions directly to one of the TAs (Kostas or Okke). Please type your solutions if possible and feel free to use the LaTeX template provided on the course home page. Email your solutions to `cs364a-aut1314-submissions@cs.stanford.edu`. If you prefer to hand-write your solutions, you can give it to one of the TAs in person.

(3) If you don't solve a problem to completion, write up what you've got: partial proofs, lemmas, high-level ideas, counterexamples, and so on.

(4) Except where otherwise noted, you may refer to your course notes, and to the textbooks and research papers listed on the course Web page *only*. You cannot refer to textbooks, handouts, or research papers that are not listed on the course home page. If you do use any approved sources, make you sure you cite them appropriately, and make sure that all your words are your own.

(5) You can discuss the problems verbally at a high level with other groups. And of course, you are encouraged to contact the course staff (via Piazza or office hours) for additional help.

(6) No late assignments will be accepted.

Problem 14

(10 points) This problem concerns the same class of reverse auctions as in Exercise 34. Under the same assumptions as in that exercise, prove that the corresponding DSIC reverse auction is in fact *weakly group-strategyproof*. This means that for every *coalition* $S \subseteq B$ of bidders, every set $\mathbf{b}_{-S}$ of bids of the other bidders $B \setminus S$, and every set $\mathbf{v}_S$ of true valuations for S, there is no set $\mathbf{b}_S$ of bids that results in every bidder of S receiving strictly higher utility than with truthful bids $\mathbf{v}_S$.

Problem 15

This problem considers auctions that provide revenue guarantees.

(a) (3 points) Consider an unlimited-supply auction (n bidders with private valuations, n identical goods, each bidder wants only one) with a twist: the auctioneer incurs a fixed production cost of 1 if there is at least one winner; if no goods are sold, then no such cost is incurred. Call an auction for this problem *budget-balanced* if, whenever there is at least one winner, the prices charged to the winners sum to exactly the cost incurred (namely, 1). Define the *surplus* of an outcome with winners S to be 0 if $S = \emptyset$ and $-1 + \sum_{i \in S} v_i$ otherwise.

Note that the surplus can be maximized in this problem using the extension of the VCG mechanism described in Problem 12(a). Prove that with the standard VCG payments (in which losers pay 0), this

VCG mechanism is not budget-balanced — in fact, it can generate 0 revenue even when the auctioneer incurs cost 1.

(b) (5 points) Prove that the following direct-revelation mechanism (given bids $\mathbf{b}$) is budget-balanced and DSIC:

- Initialize S to be all bidders.
- While $S \neq \emptyset$:
  - If $b_i \geq \frac{1}{|S|}$ for every $i \in S$, then halt. The winning bidders are those in S and each pays $\frac{1}{|S|}$.
  - Otherwise, delete from S an arbitrary bidder with $b_i < \frac{1}{|S|}$.
- If S becomes empty, then halt with no winners (and no payments).

(c) (8 points) The mechanism in (b) does not generally maximize the surplus. Precisely, show that the largest-possible difference (over all possible valuation profiles $\mathbf{v}$) between the maximum surplus and the surplus achieved by this mechanism is exactly $-1 + \sum_{i=1}^{n} \frac{1}{i}$. (This is roughly $\ln n$, minus a small constant.)

(d) (5 points) We can generalize the result in (b) as follows. Consider an unlimited-supply auction with players N, in which the auctioneer incurs a (publicly known) cost of $C(S)$ when the set of winners is $S \subseteq N$. Assume that $C(\emptyset) = 0$, that C is nondecreasing (meaning $C(S) \leq C(T)$ whenever $S \subseteq T$), and that C is *submodular*, meaning that

$$C(T \cup \{i\}) - C(T) \leq C(S \cup \{i\}) - C(S)$$

whenever $S \subseteq T$ and $i \notin T$.[1]

The *Shapley value of i in S*, denoted $\chi_{Sh}(i, S)$, is defined as follows. For an ordering π of the players of S, let T_π denote those preceding i in π. Then $\chi_{Sh}(i, S) := \mathbf{E}_\pi[C(T_\pi \cup \{i\}) - C(T_\pi)]$, where π is chosen uniformly at random. In other words, assuming that the players of S are added to the empty set 1-by-1 in a random order, $\chi_{Sh}(i, S)$ is the expected jump in cost caused by i's arrival.

Prove that, under the assumptions on C above, $\chi_{Sh}(i, S) \geq \chi_{Sh}(i, T)$ whenever $S \subseteq T$.

(e) (4 points) By using Shapley values as prices, generalize the budget-balanced truthful mechanism in (b) to an unlimited-supply auction with an arbitrary nondecreasing, submodular cost function. Be sure to prove that your mechanism is DSIC and budget-balanced.

(f) (5 points) Prove that the mechanism in (e) is weakly groupstrategyproof (see Problem 14 for a definition).

Problem 16

In this problem we modify the multi-item auction setting of the clinching auction (Lecture 9) in two ways. First, we make the problem easier by assuming that bidders have no budgets. Along a different axis, we make the problem more general: rather than having a common value v_i for every good that it gets, a bidder i has a private marginal valuation v_{ij} for its jth good, given that it already has $j-1$ goods. (Previously, v_{ij} was independent of j.) Thus, if i receives k goods at a combined price of p, its utility is $(\sum_{j=1}^{k} v_{ij}) - p$. We assume throughout this problem that every bidder gets diminishing returns from goods: for every bidder i, $v_{i1} \geq v_{i2} \geq v_{i3} \geq \cdots \geq v_{im}$.

(a) (3 points) The VCG mechanism has a fairly simple explicit description in this setting. What is it?

[1] This is a set-theoretic type of "diminishing returns". For example, when $C(S)$ depends only on $|S|$, submodularity becomes discrete concavity.

(b) (7 points) Suppose we adapt the clinching auction from lecture to the present setting, by redefining bidder demand functions in the obvious way. That is, for a bidder i that has already clinched ℓ goods, and a price p, we define $D_i(p)$ as $\max\{k - \ell, 0\}$, where $k \leq m$ is the largest value of j with $v_{ij} > p$.[2]

Prove that the allocation and payments of this clinching auction coincide with that of the VCG mechanism.

Problem 17

(10 points) Prove that the Gale-Shapley proposal algorithm is DSIC for the men — i.e., reporting a false total ordering over the women can only cause a man to be matched to a woman ranked lower in his (true) preference list.

Problem 18

(10 points) *Algorithmic Game Theory*, Exercise 18.2(b).

Problem 19

(15 points) *Algorithmic Game Theory*, Exercise 18.8(e).

Problem 20

In this problem we consider nonatomic selfish routing networks with one source, one sink, one unit of selfish traffic, and affine cost functions (of the form $c_e(x) = a_e x + b_e$ for $a_e, b_e \geq 0$). In parts (a)-(c), we consider the objective of the *maximum cost* incurred by a flow f:

$$\max_{P\,:\,f_P>0} \sum_{e \in P} c_e(f_e).$$

The *price of anarchy* is then defined in the usual way, as the ratio between the maximum cost of an equilibrium flow and that of a flow with minimum-possible maximum cost. (Of course, in an equilibrium flow, all traffic incurs exactly the same cost; this is not generally true in a non-equilibrium flow.)

(a) (4 points) Prove that in a network of parallel links (each directly connecting the source to the sink), the price of anarchy with respect to the maximum cost objective is 1.

(b) (4 points) Prove that the price of anarchy with respect to the maximum cost objective can be as large as 4/3 in general networks (with affine cost functions, one source and one sink).

(c) (5 points) Prove that the price of anarchy with respect to the maximum cost objective is never larger than 4/3 (in networks with affine cost functions, one source and one sink).

[Hint: try to reduce this to facts you already know.]

(d) (7 points) A flow that minimizes the average cost of traffic generally routes some traffic on costlier paths than others. Prove that the ratio between the cost of the longest used path and that of the shortest used path in a minimum-cost flow is at most 2 (in networks with affine cost functions, one source and one sink). Prove that this bound can be achieved.

[2] As in the version of the auction in lecture, we can count goods j with $v_{ij} = p$ toward i's demand or not, according to convenience. Alternatively, feel free to assume that all of the v_{ij}'s are distinct for this problem.

Problem 21

Recall the Generalized Second Price (GSP) keyword auction from Problem 3. For this problem we'll think about bidders with known valuations and the pure-strategy Nash equilibria of the corresponding game. Note that in part (e) of Problem 3 you proved that in every such game (for any k, n, valuations, and click-through rates), there is a pure-strategy Nash equilibrium with the maximum-possible surplus. That is, the best Nash equilibrium captures the full surplus. This problem investigates additional Nash equilibria that do not share this property.

(a) (5 points) Show that even when $k = 1$ and $n = 2$, the price of anarchy of the GSP game can be arbitrarily bad.

(b) (5 points) Consider now a Nash equilibrium in which every bid b_i is at most the player's valuation v_i. Suppose that players i and j are assigned to slots with click-through rates α_h and α_ℓ, respectively, with $h < \ell$. Prove that

$$\frac{\alpha_\ell}{\alpha_h} + \frac{v_i}{v_j} \geq 1.$$

(c) (10 points) Consider again a Nash equilibrium in which every bid b_i is at most the player's valuation v_i. Prove (perhaps using (b)) that the surplus of this equilibrium is at least 50% of the maximum possible.