Set Theory 292B: An Ideal Characterization of Mahlo Cardinals

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Abstract

In this term paper we show an ideal characterization of Mahlo cardinals; a cardinal κ is (strongly) Mahlo if and only if there exists a nontrivial κ-complete κ-normal ideal on it. It is a summary of one part of works in [1], [2].

1 Preliminary

In this paper we use κ to denote a regular uncountable cardinal unless the opposite is stated. An *ideal* I on κ is a subset of κ with the following properties:

(1) $\emptyset \in I$ but $\kappa \notin I$.

(2) $A \cup B \in I$ if $A, B \in I$.

(3) $B \in I$ if $B \subseteq A$ and $A \in I$.

An ideal I is *nontrivial* if $I \neq \{\emptyset\}$ and for each $\alpha \in \kappa$, $\{\alpha\} \in I$[1]. Similarly we have a dual concept called *filter*. A *filter* F on κ is a subset of κ such that:

(1) $\emptyset \notin I$ but $\kappa \in F$.

(2) $A \cap B \in F$ if $A, B \in I$.

(3) $B \in F$ if $B \supseteq A$ and $A \in F$.

It is easily seen that if X is an ideal (resp. filter) on κ, then the set $\{A \subseteq \kappa : \ \kappa - A \in X\}$ is a filter (resp. ideal) on κ. We denote the ideal (resp. filter) dual of X by X^*. Also we write X^- for the set $\{A \subseteq \kappa : \ A \notin X\}$. An ideal I is κ-complete if for any collection $\{A_\alpha : \alpha < \lambda\}$ where $A_\alpha \in I$ and $\lambda < \kappa$,

$$\bigcup_{\alpha<\lambda} A_\alpha \in I$$

[1]Note that standard nontriviality definition only requires $I \neq \{\emptyset\}$.

Similarly, a filter F is κ-complete if for any collection $\{A_\alpha : \alpha < \lambda\}$ where $A_\alpha \in F$ and $\lambda < \kappa$,

$$\bigcap_{\alpha<\lambda} A_\alpha \in F$$

Obviously if I is the dual of F, then I is κ-complete iff F is κ-complete.

A set $C \subseteq \kappa$ is *closed unbounded* (*club*) if $\sup C = \kappa$ and any increasing sequence $\langle \alpha_\eta : \eta < \lambda \rangle$ of elements of C with $\lambda < \kappa$ is closed, i.e., $\sup\{\alpha_\eta : \eta < \lambda\} \in C$. We use **Club** to denote the class of all clubs on κ. A subset S of κ is *stationary* if for any *club* C, $S \cap C \neq \emptyset$.

A function f on $X \subseteq \kappa$ is *regressive* if $f(\alpha) < \alpha$ for any $\alpha \in \kappa$, $\alpha \neq 0$. An ideal I is *normal* if for each $X \in I^-$ for each regressive function $f : X \to \kappa$ there exists $Y \subseteq X$, $Y \in I^-$ such that f is constant on Y.

We can generalize the notions of normal ideal and regressive function to κ-normal ideal and κ-regressive function respectively. Let $[\kappa]^{<\kappa}$ denote the set of increasing sequences of length less than κ. For $X \subseteq \kappa$, let $\overline{X}$ denote the order type of X. A function $f : X \to [\kappa]^{<\kappa}$ is *κ-regressive* if $f(\alpha) \subseteq \alpha$ and $\overline{f(\alpha)} < \alpha$ for any $\alpha < \kappa$, $\alpha > 0$. An ideal I is *κ-normal* if for each $X \in I^-$ for each κ-regressive function $f : X \to [\kappa]^{<\kappa}$ there exists a set $Y \subseteq X$, $Y \in I^-$ such that f is constant on Y. Obviously every κ-normal ideal is normal.

Let $D = \langle D_\alpha : \alpha < \kappa \rangle$ be a κ-sequence of subsets of κ. A *diagonal intersection* ΔD of D is defined as follows:

$$\Delta D = \{\alpha < \kappa : \alpha \in \bigcap_{\xi<\alpha} D_\xi\}$$

Let F be a filter on κ and D be any κ-sequence of elements in F. F is a *normal* if F is closed under such diagonal intersection, i.e., $\Delta D \in F$.

Let κ be an uncountable cardinal. Let **Reg** denote the set of all regular cardinals less than κ. κ is (strongly) *Mahlo* if it is (strongly) inaccessible and **Reg** is stationary.

2 16 Easy Lemmas

In this section we list 16 easy lemmas that pave the way toward the main theorems. Proofs of obvious or well-known results are omitted. Again by default κ is a regular uncountable cardinal.

Lemma 2.1. *(1) If C_1, C_2 are clubs on κ, then $C_1 \cap C_2$ is a club.*

(2) The intersection of less than κ clubs is a club.

(3) If C is a club and S is a stationary set, then $C \cap S$ is stationary.

*(4) Let $C = \langle C_\alpha : \alpha < \kappa \rangle$ be sequence of clubs. ΔC is a club. In other words, **Club** is closed under diagonal intersection.*

Lemma 2.2. *Let $\boldsymbol{C}_\kappa$ be the set $\{A \subseteq \kappa : \exists C \in \mathbf{Club}\ C \subseteq A\}$. $\boldsymbol{C}_\kappa$ is a κ-complete filter on κ.*

Lemma 2.3. *Let $\boldsymbol{NS}_\kappa$ be the ideal dual of $\boldsymbol{C}_\kappa$. $\boldsymbol{NS}_\kappa$ consists of all nonstationary subsets of κ and is normal.*

Lemma 2.4. *An ideal I over κ is normal iff its filter dual F is norml.*

Lemma 2.5. *Let F be a normal filter on κ that contains all final segments of κ, then F contains $\boldsymbol{C}_k$.*

Proof. cf. [3], page 60. □

Lemma 2.6. *Let $X \subseteq \kappa$ and function $f : X \to [\kappa]^{<\kappa}$ is κ-regressive. Let $g : X \to \kappa$ defined by $g(\alpha) = \sup f(\alpha)$. Then g is regressive.*

Lemma 2.7. *Let $A = \langle A_\alpha : \alpha < \kappa \rangle$, $B = \langle B_\alpha : \alpha < \kappa \rangle$ be sequences of subsets of κ such that $A_\alpha \supseteq B_\alpha \cap C$ for $\alpha < \kappa$, $C \subseteq \kappa$. Then $\Delta A \supseteq \Delta B \cap C$.*

Proof.

$$\begin{aligned} \alpha \in \Delta B \cap C &\Leftrightarrow \alpha \in C \text{ and for any } \xi < \alpha, \alpha \in B_\xi \\ &\Rightarrow \alpha \in C \text{ and for any } \xi < \alpha, \alpha \in A_\xi \\ &\Leftrightarrow \alpha \in C \cap \Delta A \end{aligned}$$

□

Lemma 2.8. *Let κ be a Mahlo cardinal. Let $\mathbf{C}_\kappa[\mathbf{Reg}] = \{A \subseteq \kappa : \exists C \in \mathbf{C}_\kappa \ A \supseteq C \cap \mathbf{Reg}\}$. $\mathbf{C}_\kappa[\mathbf{Reg}]$ is a κ-complete normal filter on κ.*

Proof. (1) $\mathbf{C}_\kappa[\mathbf{Reg}]$ is a filter. It is obvious that $\kappa \in \mathbf{C}_\kappa[\mathbf{Reg}]$ and if $A \in \mathbf{C}_\kappa[\mathbf{Reg}]$ and $A \subseteq B$ then $B \in \mathbf{C}_\kappa[\mathbf{Reg}]$. Let $A, B \in \mathbf{C}_\kappa[\mathbf{Reg}]$. Then there exists $C_1, C_2 \in \mathbf{C}_\kappa$ such that $A \supseteq C_1 \cap \mathbf{Reg}$ and $B \supseteq C_2 \cap \mathbf{Reg}$. So $A \cap B \supseteq C_1 \cap C_2 \cap \mathbf{Reg}$. By Lemma 2.1 $C_1 \cap C_2 \in \mathbf{C}_\kappa$ and hence $A \cap B \in \mathbf{C}_\kappa[\mathbf{Reg}]$.

(2) $\mathbf{C}_\kappa[\mathbf{Reg}]$ is κ-complete. Let $\{A_\alpha : \alpha < \lambda\}$ be a family of subsets of $\mathbf{Reg}$ where $\lambda < \kappa$. There exists a family $\{C_\alpha : \alpha < \lambda\}$ of subsets of $\mathbf{C}_\kappa$ such that $A_\alpha \supseteq C_\alpha \cap \mathbf{Reg}$ for $\alpha < \lambda$. Then

$$\bigcap_{\alpha<\lambda} A_\alpha \supseteq \bigcap_{\alpha<\lambda} (C_\alpha \cap \mathbf{Reg}) = \bigcap_{\alpha<\lambda} C_\alpha \cap \mathbf{Reg}$$

It follows from κ-completeness (Lemma 2.2) of $\mathbf{C}_\kappa$ that $\mathbf{C}_\kappa[\mathbf{Reg}]$ is κ-complete.

(3) $\mathbf{C}_\kappa[\mathbf{Reg}]$ is normal. Let $A = \langle A_\alpha : \alpha < \kappa \rangle$ be a sequence of sets in $\mathbf{C}_\kappa[\mathbf{Reg}]$. We have sequence $C = \langle C_\alpha : \alpha < \kappa \rangle$ such that $C_\alpha \cap \mathbf{Reg} \subseteq A_\alpha$ for $\alpha < \kappa$. By Lemma 2.7 $\Delta A \supseteq \Delta C \cap \mathbf{Reg}$. Since $\mathbf{C}_\kappa$ is normal, $\Delta C \in \mathbf{C}_\kappa$. So $\mathbf{C}_\kappa[\mathbf{Reg}]$ is closed under diagonal intersection and hence is normal. □

Lemma 2.9. *Any nontrivial κ-complete ideal I contains all bounded segments and hence contains no final segments.*

Proof. Let $X \subseteq \kappa$ and $\alpha = \sup X$. Since X is bounded, $\alpha < \kappa$. By nontriviality, $\{\eta\} \in I$ for all $\eta \in \kappa$. Since

$$\alpha = \bigcup \{\{\eta\} : \eta \in \alpha\}$$

By κ-complete $\alpha \in I$. So $X \in I$ as $X \subseteq \alpha$. Also $\kappa - \alpha \notin I$, for otherwise $\kappa \in I$. □

Lemma 2.10. *Let κ be an uncountable singular cardinal and I be nontrivial κ-complete ideal on it. If X is unbounded and $|X| < \kappa$, then $X \notin I$.*

Proof. Suppose $X \in I$. For each $\alpha \in X$, $\{\alpha\} \in I$. Since $|X| < \kappa$ and

$$\bigcup \{\{\alpha\} : \alpha \in X\} = \sup X = \kappa$$

by κ-completeness $\kappa \in I$, a contradiction. □

Lemma 2.11. *Let κ be an uncountable cardinal for which there exists a κ-complete κ-normal ideal I. Then κ is regular.*

Proof. Suppose κ is not regular. Let $\lambda = cf(\kappa)$. Let $B = \langle \alpha_\xi : \xi < \lambda \rangle$ be a λ-sequence cofinal with κ. Without loss of generality, assume that $\alpha_\xi > \lambda$ for each $\xi < \lambda$. Since $|B| = \lambda < \kappa$, by Lemma 2.10 $B \notin I$. Then $B \in I^-$. Define a function $f : B \to [\kappa]^{<\kappa}$ by $f(\alpha_\xi) = \xi$. Since $\xi < \lambda < \alpha_\xi$, f is one-to-one and κ-regressive. This contradicts κ-normality. $\square$

Lemma 2.12. *Let κ be an uncountable cardinal for which there exists a κ-complete κ-normal ideal I. Then κ is a strong limit cardinal.*

Proof. Suppose κ is not a strong limit cardinal. Then there exists $\lambda < \kappa$ such that $2^\lambda > \kappa$. Define $f : \kappa \to 2^\lambda$ to be the one-to-one function. Let $A = \{\alpha < \kappa : \alpha > \lambda\}$. Since A is a final segment, by Lemma 2.9 $A \notin I$, and so $A \in I^-$. Consider $g = f \restriction A$. For any $\alpha \in A$, $\alpha > \lambda$ and $f(\alpha) \subseteq \lambda$. So $\overline{f(\alpha)} < f(\alpha)$, i.e., f is κ-regressive, contradicting κ-normality. $\square$

Lemma 2.13. *Let η be a cardinal such that $2^\eta < \kappa$. Let X be stationary subset of κ and $f : X \to \mathcal{P}(\eta)$. Then there exists a stationary set $Y \subseteq X$ such that f is constant on Y.*

Proof. Let sequence $A = \langle A_\alpha : \alpha < 2^\eta \rangle$ enumerates all subsets of η. Let $B_\alpha = f^{-1}[A_\alpha]$, the inverse image of A_α under f. Suppose none of B_α is stationary. Then there exists a sequence of clubs $C = \langle C_\alpha : \alpha < 2^\eta \rangle$ such that $C_\alpha \cap A_\alpha = \emptyset$ for each $\alpha < 2^\eta$. Since $\mathbf{C}_\kappa$ is κ-complete and $2^\eta < \kappa$,

$$\bigcap C = \bigcap_{\alpha < 2^\eta} C_\alpha \in \mathbf{Club}$$

Let $D = \bigcap C \cap X$. Since X is stationary, $D \neq \emptyset$. But for any $\xi \in D$ $f(\xi) \neq A_\alpha$ for all $\alpha < 2^\eta$. That is, $f(\xi) \notin \mathcal{P}(\eta)$, a contradiction. $\square$

Lemma 2.14. *Define $B_\xi = \{\alpha < \kappa : cf(\alpha) > \xi \vee \alpha \leq \xi\}$ for each $\xi < \kappa$. (Again we assume $cf(\alpha) = 0$ if α is not a limit ordinal.) Let $B = \langle B_\xi : \xi < \kappa \rangle$ be the enumerating sequence. Then $\Delta B =$ **Reg**.*

Proof.

$$\begin{aligned} \alpha \in B &\Leftrightarrow \alpha \in B_\xi \text{ for all } \xi < \alpha \\ &\Leftrightarrow cf(\alpha) > \xi \text{ or } \alpha \leq \xi \textit{ for all } \xi < \alpha \\ &\Leftrightarrow cf(\alpha) > \xi \textit{ for all } \xi < \alpha \\ &\Leftrightarrow \alpha \in \mathbf{Reg} \end{aligned}$$

$\square$

Lemma 2.15. *Let I be a κ-normal ideal on κ. For each $\xi < \kappa$ define $B^-_\xi = \{\alpha < \kappa : cf(\alpha) \leq \xi < \alpha\}$. Then $B^-_\xi \in I$ for all $\xi < \kappa$.*

Proof. Let $\xi < \kappa$. Suppose $B^-_\xi \notin I$ and then $B^-_\xi \in I^-$. Define a function $f : B^-_\xi \to [\kappa]^{<\kappa}$ such that $f(\alpha) = \beta$ if $\alpha = \beta + 1$ for some β, and otherwise $f(\alpha)$ is a sequence cofinal with α with length of $cf(\alpha)$. First note that there is no regular cardinal in B^-_ξ. Second $\overline{f(\alpha)} = cf(\alpha) < \alpha$ if α is non-regular limit ordinal and $\overline{f(\alpha)} = \alpha - 1 < \alpha$ if α is a non-limit ordinal. So f is κ-regressive. It is easily seen that f is one-to-one. A contradiction. $\square$

Lemma 2.16. *Let I be a κ-normal ideal on κ. Then $\textbf{Reg} \in I^*$.*

Proof. Let B_η, B and B^-_η be as defined in Lemma 2.14 and 2.15. By Lemma 2.15 $B^-_\eta \in I$. So $B_\eta \in I^*$. Since I^* is a κ-normal and then normal, $\Delta B \in I^*$. By Lemma 2.14 $\mathbf{Reg} \in I^*$. $\square$

3 Main Theorems

Theorem 3.1. *If κ is Mahlo cardinal then there exists a nontrivial κ-complete κ-normal ideal on κ.*

Proof. Let $\mathbf{C}_\kappa[\mathbf{Reg}]$ be as defined in Lemma 2.8. By that lemma $\mathbf{C}_\kappa[\mathbf{Reg}]$ is a κ-complete normal filter. Let $\mathbf{I}_\kappa[\mathbf{Reg}]$ be its ideal dual. By Lemma 2.4 $\mathbf{I}_\kappa[\mathbf{Reg}]$ a κ-complete and normal. It suffices to show that it is also κ-normal. Let

$$\mathbf{I}^-_\kappa[\mathbf{Reg}] = \{A \subseteq \kappa : A \notin \mathbf{I}_\kappa[\mathbf{Reg}]\}$$

Since

$$\begin{aligned} \mathbf{I}_\kappa[\mathbf{Reg}] &= \{A \subseteq \kappa : \kappa - A \in \mathbf{C}_\kappa[\mathbf{Reg}]\} \\ &= \{A \subseteq \kappa : \exists C \in \mathbf{C}_\kappa[\mathbf{Reg}]\ \kappa - A \supseteq C \cap \mathbf{Reg}\} \end{aligned}$$

we have

$$\begin{aligned} \mathbf{I}^-_\kappa[\mathbf{Reg}] &= \{A \subseteq \kappa : \forall C \in \mathbf{C}_\kappa[\mathbf{Reg}]\ \kappa - A \not\supseteq C \cap \mathbf{Reg}\} \\ &= \{A \subseteq \kappa : \forall C \in \mathbf{C}_\kappa[\mathbf{Reg}]\ \exists c \in C \cap \mathbf{Reg} c \in A\} \\ &= \{A \subseteq \kappa : \forall C \in \mathbf{C}_\kappa[\mathbf{Reg}]\ C \cap \mathbf{Reg} \cap A \neq \emptyset\} \end{aligned}$$

Note that for each $X \subseteq \kappa$ $X \in \mathbf{I}^-_\kappa[\mathbf{Reg}]$ if and only if both X and $X \cap \mathbf{Reg}$ are stationary. Now let $A \in \mathbf{I}^-_\kappa[\mathbf{Reg}]$ and $S = A \cap \mathbf{Reg}$. S is stationary. Let $f : A \to [\kappa]^{<\kappa}$ be κ-regressive. By the definition of κ-regressiveness $g = f \restriction S$ is κ-regressive too. Define $h : S \to \kappa$ by $h(\alpha) = \sup g(\alpha)$ for $\alpha \in S$. By Lemma 2.6 h is regressive. Since $\mathbf{I}^-_\kappa[\mathbf{Reg}]$ is normal, there exists a stationary set $S' \subseteq S$ and $\eta < \kappa$ such that h is constant on S', i.e., $h(\alpha) = \eta$ for $\alpha \in S'$. So $g(\alpha) \in \mathcal{P}(\eta)$ for $\alpha \in S'$. By Lemma 2.13 there exists a stationary set $S'' \subseteq S'$ such that g is constant on S''. Then f is constant on S''. Obviously $S'' \in \mathbf{I}^-_\kappa[\mathbf{Reg}]$ since $S'' \subseteq \mathbf{Reg}$ and S'' is stationary. Hence $\mathbf{I}^-_\kappa[\mathbf{Reg}]$ is κ-normal. $\square$

Theorem 3.2. *An uncountable cardinal κ with a nontrivial κ-complete κ-normal ideal is Mahlo.*

Proof.

(1) κ is regular. By Lemma 2.11.

(2) κ is a strong limit. By Lemma 2.12.

(3) $\mathbf{Reg}$ is stationary. By Lemma 2.16 $\mathbf{Reg} \in I^*$. By Lemma 2.9 I contains all bounded sets of κ and hence I^* contains all final segments of κ. Also since I^* is normal, by Lemma 2.5 I^* contains all clubs. So for any club C, $\mathbf{Reg} \cap C \in I^*$, and hence $\mathbf{Reg} \cap C \neq \emptyset$.

$\square$

Theorem 3.3. *An uncountable cardinal κ is Mahlo if and only if there exists a nontrivial κ-complete κ-normal ideal over it.*

Proof. By Theorem 3.1 and 3.2. $\square$

References

[1] Qi Feng, **An Ideal Characterization of Mahlo Cardinals** Journal of Symbolic Logic, vol. 54 (1989), pp. 467-473, Association for Symbolic Logic.

[2] Thomas Jech, **Some Combinatorial Problems Concerning Uncountable Cardinals**, Annals of Mathematical Logic, vol 5 (1973), pp. 165-198, North-Holland Publishing Company.

[3] Thomas Jech, **Set Theory**, 2nd ed., 1997, Springer-Verlag.