Homework 1

1. Prove that if X and Y are random variables then $\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y]$.

   Solution: Proof 1:

$$\begin{aligned}
\mathrm{E}[X+Y] &= \sum_v v \cdot \Pr[X+Y=v] \\
&= \sum_v \Pr[X=v] \left(\sum_w \Pr[Y=w|X=v] \cdot (v+w) \right) && (1)\\
&= \sum_v \Pr[X=v] \cdot v \cdot \left(\sum_w \Pr[Y=w|X=v] \right) + \sum_v \Pr[X=v] \left(\sum_w \Pr[Y=w|X=v] \cdot w \right) \\
&= \sum_v \Pr[X=v] \cdot v \cdot 1 + \sum_w \sum_v \Pr[Y=w|X=v] \Pr[X=v] \cdot w && (2)\\
&= \sum_v v \cdot \Pr[X=v] + \sum_w w \cdot \Pr[Y=w] && (3)\\
&= \mathrm{E}[X] + \mathrm{E}[Y]
\end{aligned}$$

   Try to verify for yourself that the identities used in (1), (2) and (3) make sense.

   Proof 2:

$$\begin{aligned}
\mathrm{E}[X+Y] &= \sum_v v \cdot \Pr[X+Y=v] \\
&= \sum_v \sum_w (v+w) \cdot \Pr[X=v \wedge Y=w] && (4)\\
&= \sum_v \sum_w v \cdot \Pr[X=v \wedge Y=w] + \sum_v \sum_w w \cdot \Pr[X=v \wedge Y=w] \\
&= \sum_v v \sum_w \Pr[X=v \wedge Y=w] + \sum_w w \sum_v \Pr[X=v \wedge Y=w] && (5)\\
&= \sum_v v \cdot \Pr[X=v] + \sum_w w \Pr[Y=w] && (6)\\
&= \mathrm{E}[X] + \mathrm{E}[Y]
\end{aligned}$$

   Try to verify for yourself that the identities used in (4), (5) and (6) make sense. These proofs are really the same, since after all $\Pr[X=v \wedge Y=w] = \Pr[X=v]\Pr[Y=w|X=v]$ (why?).

2. When throwing n balls into n bins, what is the expected number of bins with *exactly* one ball? Compute an exact formula and, if applicable, an asymptotic approximation.

Solution: Let X be the random variable which is the number of bins with exactly one ball. Let X_i be the indicator random variable for whether bin i has exactly one ball. Then $X = \sum_{i=1}^{n} X_i$, and $\mathrm{E}[X] = \mathrm{E}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \mathrm{E}[X_i]$. Since all bins are identical as usual, we can think only about $\mathrm{E}[X_1]$, and multiply it by n to get the final figure.

Now $\mathrm{E}[X_1] = \Pr[X_1 = 1] = \frac{n\cdot(n-1)^{n-1}}{n^n}$. The numerator is the number of ways to put exactly one ball in bin 1, since there are n choices for the single ball to fall into bin 1, and for each such choice, $(n-1)^{n-1}$ choices for the other balls to fall into the other bins. The denominator is the number of ways to put n balls into n bins. So $\mathrm{E}[X]$ is

$$\mathrm{E}[X] = n\frac{n\cdot(n-1)^{n-1}}{n^n} = n\frac{(1-\frac{1}{n})^n}{1-1/n} \sim \frac{n}{e}$$

3. Consider the following probabilistic process.

   Set $c = 0$
   Repeat n times
       Toss a fair coin
           If heads then $c = c + 1$
           else $c = c - 1$.

   The sample space here is $\{H, T\}^n$, the set of all possible outcomes when tossing a coin n times, and each event has probability $1/2^n$.

   Let X be the random variable defined as the value of c at the end of the process.

   (a) Compute $\mathbf{E}[X]$.
   Solution: By symmetry, it must be that $\mathrm{E}[X] = 0$.
   If you don't like that solution, here's another one: write $X = \sum_{i=1}^{n} X_i$, where $X_i = 1$ if the coin comes up heads, and $X_i = -1$ otherwise. Then

   $$\mathrm{E}[X] = \sum_{i=1}^{n} \mathrm{E}[X_i] = \sum_{i=1}^{n} 0 = 0$$

   since $\mathrm{E}[X_i] = -1(1/2) + 1(1/2) = 0$. The second proof is more useful because it will still work with a biased coin flip, or even when we increment and decrement by different amounts.

   (b) For even n, show that $\mathbf{Pr}[X = 0] \sim \sqrt{2/(\pi n)}$. What about odd n?
   Note: you will need to use Stirling's approximation, which says that

   $$n! \sim \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n}\ .$$

   Solution: The probability that $X = 0$ is the probability that the number of heads equals the number of tails. This is impossible if n is odd. If n is even, there are $\binom{n}{n/2}$ ways to have the number of heads equal the number of tails, and there are 2^n total ways to toss the coins. Hence

$$\begin{aligned}
\Pr[X=0] &= \frac{\binom{n}{n/2}}{2^n} \\
&= \frac{\frac{n!}{(n/2)!(n/2)!}}{2^n} \\
&\sim \frac{\frac{(n/e)^n\sqrt{2\pi n}}{((\frac{n}{2e})^{n/2}\sqrt{\pi n})^2}}{2^n} \\
&= \frac{\sqrt{2\pi n}}{\pi n} \\
&= \sqrt{2/(\pi n)}
\end{aligned}$$