

Homework 1

1. Prove that if X and Y are random variables then $\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y]$.

2. When throwing n balls into n bins, what is the expected number of bins with *exactly* one ball? Compute an exact formula and, if applicable, an asymptotic approximation.

3. Consider the following probabilistic process.

    Set $c = 0$
    Repeat n times
        Toss a fair coin
            If heads then $c = c + 1$
            else $c = c - 1$.

    The sample space here is $\{H, T\}^n$, the set of all possible outcomes when tossing a coin n times, and each event has probability $1/2^n$.

    Let X be the random variable defined as the value of c at the end of the process.

    (a) Compute $\mathbf{E}[X]$.

    (b) For even n, show that $\mathbf{Pr}[X = 0] \sim \sqrt{2/(\pi n)}$. What about odd n?
    Note: you will need to use Stirling's approximation, which says that

    $$n! \sim \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n} \ .$$