Homework 2

1. Consider again the experiment in which we toss m labeled balls at random into n labeled bins, and let the random variable X be the number of empty bins. We have seen in Note 1 that $\mathbf{E}[X] = n\left(1 - \frac{1}{n}\right)^m$.

   (a) By writing $X = \sum_i X_i$ for suitable random variables X_i, show how to derive the following exact formula for the variance of X:

   $$\mathbf{Var}[X] = n\left(1 - \frac{1}{n}\right)^m + n(n-1)\left(1 - \frac{2}{n}\right)^m - n^2\left(1 - \frac{1}{n}\right)^{2m}.$$

   (b) What is the asymptotic value (as $n \to \infty$) of $\mathbf{Var}[X]$ in the cases (i) $m = n$ and (ii) $m = n \ln n$?

2. You are given a biased coin which comes up heads with probability p and tails with probability $1 - p$. What is the expected number of times you need to toss the coin before the first head appears (including the head itself)?

3. Consider, hypothetically, a *fair game* in a casino: on each play, you may stake any amount \$$S$; you win or lose with probability 1/2 each (all plays being independent); if you win you get your stake back plus \$$S$; if you lose you lose your stake.

   (a) What is the expected number of plays before your first win (including the play on which you win)?

   (b) The following gambling strategy, known as the "martingale," was popular in European casinos in the 18th century: on the first play, stake \$1; on the second play \$2; on the third play \$4; on the kth play \$$2^{k-1}$. Stop (and leave the casino!) when you first win.

   (c) Show that, if you follow the martingale strategy, you will leave the casino \$1 richer with probability 1. (Maybe this is why the strategy is banned in most modern casinos.)

   (d) To discover the catch in this seemingly infallible strategy, let X be the r.v. that measures your maximum loss before winning (i.e., the amount of money you have lost before the play on which you win). Show that $\mathbf{E}[X] = \infty$. What does this imply about your ability to play the martingale strategy in practice?