Notes for Lecture 9

1 More on Eigenvalues and Expanders

Recall the definition of (normalized) edge expansion.

Definition 1 (Edge-expansion of a graph) *The edge-expansion of a graph G is defined as*

$$h(G) := \min_{|S| \leq |V|/2} \frac{edges(S, V-S)}{d|S|}$$

Let $G = (V, E)$ be a d-regular graph, fixed for the rest of this section, and $\lambda_1 \geq \cdots \geq \lambda_n$ be its eigenvalues with multiplicites, and $x_1, \ldots, x_n$ be a corresponding set of orthnormal eigenvectors. We have $\lambda_1 = 1$ and $x_1 = \frac{1}{\sqrt{n}}(1, \ldots, 1)$.

We proved that

$$\lambda_2 = \max_{x \perp \mathbf{1}} \frac{xMx^t}{xx^t} \tag{1}$$

and we also observed that for every real vector $x \in \mathbb{R}^n$

$$\sum_{u,v} M(u,v)(x(u) - x(v))^2 = 2xx^T - 2xMx^T \tag{2}$$

and, combining the two, we have another equivalent characterization of λ_2.

$$1 - \lambda_2 = \min_{x \perp \mathbf{1}} \frac{\sum_{u,v} M(u,v)(x(u) - x(v))^2}{2xx^T} \tag{3}$$

We proved that $h(G) = 0$ if and only if $\lambda_2 = \lambda_1$. Today we look at a quantitative version of this result, that is we show that if $\lambda_1 - \lambda_2$ is large if and only if $h(G)$ is large.

Theorem 1 (Cheeger's Inequality)

$$\frac{h^2}{2} \leq 1 - \lambda_2 \leq 2h$$

2 Proof that $1 - \lambda_2 \leq 2h$

We use the same argument that established that if $\lambda_2 = 1$ then $h = 0$.

Let S be the set that achieves $h(G) = \frac{edges(S, V-S)}{d|S|}$

Let $p := |S|/|V|$ and $q := 1 - p = |V - S|/|V|$, and define the vector $x \in \mathbb{R}^n$ as $x(v) := q$ for $v \in S$ and $x(v) := p$ for $v \notin S$. By definition, x is orthogonal to $(1, \ldots, 1)$, and so, using Equation (3),

$$1-\lambda_2 \leq \frac{\sum_{u,v} M(u,v)(x(u)-x(v))^2}{2xx^T}$$

Regarding the numerator, $|x(u)-x(v)|$ is 1 when u and v are in different sides of the cut, and it is 0 otherwise, so

$$\sum_{u,v} M(u,v)(x(u)-x(v))^2 = 2 \cdot \frac{1}{d} \cdot edges(S, V-S)$$

while the denominator is

$$2 \cdot (|S| \cdot q^2 + |V-S|p^2) = 2npq^2 + 2nqp^2 = 2nqp(p+q) = 2nqp \geq np = |S|$$

and so, combining everything,

$$1-\lambda_2 \leq \frac{2edges(S,V-S)}{d|S|} = 2h$$

It is also possible to present this proof in a somewhat different form, which gives one more characterization of λ_2.

First of all, let us define another combinatorial quantity, related to edge expansion, called the *conductance* $\Phi(G)$ of a graph. For a subset S of nodes, the conductance $\Phi(G,S)$ of the cut $(S, V-S)$ is

$$\Phi(G,S) := \frac{edges(S,V-S)}{d|S| \cdot \frac{|V-S|}{|V|}}$$

intuitively, the conductance of a cut looks at the ratio between the number of edges crossing the cut compared with the average number of edges that would cross the cut in a random d-regular graph.

The conductance of a graph is the conductance of the minimal cut

$$\Phi(G) := \min_{S \subseteq V} \Phi(G,S)$$

Notice that

$$h(G) \leq \Phi(G) \leq 2h(G)$$

We will show that $1-\lambda_2 \leq \Phi(G)$ by giving a formulation of $1-\lambda_2$ as a *relaxation* of $\Phi(G)$.

We can formulate $\Phi(G)$ as the problem of optimizing over all n-bit strings representing cuts in G

$$\Phi(G) = \min_{x \in \{0,1\}^n} \frac{\frac{1}{2}\sum_{u,v} dM(u,v)|x(u)-x(v)|}{d(\sum_u x_u)(n-\sum_u x_u) \cdot \frac{1}{n}}$$

Now, notice that, for a boolean vector $x \in \{0,1\}^n$,

$$\sum_{u,v}(x(u)-x(v))^2 = 2n\sum_u x^2(u) - 2\sum_{u,v} x(u)x(v)$$

$$= 2n\sum_u x(u) - 2\sum_{u,v} x(u)x(v) = 2\left(\sum_u x_u\right)\left(n - \sum_u x_u\right)$$

and also $|x(u)-x(v)| = (x(u)-x(v))^2$; so we have

$$\Phi(G) = \min_{x\in B^n} \frac{\sum_{u,v} M(u,v)(x(u)-x(v))^2}{\frac{1}{n}\sum_{u,v}(x(u)-x(v))^2}$$

Consider now the relaxation of the problem to real vectors. Because the function we want to minimize is shift-invariant,

$$\min_{x\in\mathbb{R}^n} \frac{\sum_{u,v} M(u,v)(x(u)-x(v))^2}{\frac{1}{n}\sum_{u,v}(x(u)-x(v))^2} = \min_{x\in\mathbb{R}^n, x\perp\mathbf{1}} \frac{\sum_{u,v} M(u,v)(x(u)-x(v))^2}{\frac{1}{n}\sum_{u,v}(x(u)-x(v))^2}$$

$$= \min_{x\in\mathbb{R}^n, x\perp\mathbf{1}} \frac{\sum_{u,v} M(u,v)(x(u)-x(v))^2}{2xx^T - \frac{2}{n}\sum_{u,v} x(u)x(v)} = \min_{x\in\mathbb{R}^n, x\perp\mathbf{1}} \frac{\sum_{u,v} M(u,v)(x(u)-x(v))^2}{2xx^T} = 1-\lambda_2$$

And so we have established

$$1-\lambda_2 = \min_{x\in\mathbb{R}^n} \frac{\sum_{u,v} M(u,v)(x(u)-x(v))^2}{\frac{1}{n}\sum_{u,v}(x(u)-x(v))^2} \leq \min_{x\in\{0,1\}^n} \frac{\sum_{u,v} M(u,v)(x(u)-x(v))^2}{\frac{1}{n}\sum_{u,v}(x(u)-x(v))^2} = \Phi(G) \leq 2h$$

3 Proof that $1-\lambda_2 \geq h^2/2$

Let x_2 be the eigenvector of λ_2, assume, without loss of generality, that at most $n/2$ entries of x_2 are positive (otherwise, work with $-x_2$) and define $y\in\mathbb{R}^n$ as

$$y(v) := \max\{x_2(v), 0\}$$

We will prove the following claims

Claim 2

$$\frac{\sum_{u,v} M(u,v)\cdot(y(u)-y(v))^2}{2yy^T} \leq 1-\lambda_2$$

Claim 3

$$\sum_{u,v} M(u,v)\cdot(y(u)-y(v))^2 \geq \frac{1}{4yy^T}\cdot\left(\sum_{u,v} M(u,v)\cdot|y^2(u)-y^2(v)|\right)^2$$

Claim 4

$$\sum_{u,v} M(u,v)\cdot|y^2(u)-y^2(v)| \geq 2hyy^T$$

Combining the claims, we have

$$1-\lambda_2 \geq \frac{1}{2}\cdot\frac{\sum_{u,v}(y(u)-y(v))^2}{yy^T} \geq \frac{1}{8}\cdot\frac{\left(\sum_{u,v}|y^2(u)-y^2(v)|\right)^2}{(yy^T)^2} \geq \frac{1}{2}h^2$$

It remains to prove the claims.

The proof of the first claim is just a matter of following the definitions.

PROOF:[Of Claim 2] First, note that $yM \geq \lambda_2 y$ component-wise. Indeed, if $x_2(v) \geq 0$, then $y(v) = x_2(v)$, and so

$$yM(v) = \sum_u y(u)M(u,v) \geq \sum_u x_2(u)M(u,v) = x_2M(v) = \lambda_2 x_2(v) = \lambda_2 y(v)$$

and if $x_2(v) \leq 0$, then $y(v) = 0$ and

$$yM(v) = \sum_u y(u)M(u,v) \geq 0 = \lambda_2 y(v)$$

We also have

$$yMy^T \geq \lambda_2 yy^T$$

and so

$$\sum_{u,v}(y(u)-y(v))^2 = 2yy^T - 2yMy^T \leq 2 - 2\lambda_2$$

□

The second claim follows from Cauchy-Schwarz.

PROOF:[Of Claim 3]

$$\begin{aligned}
\sum_{u,v} M(u,v)|y^2(u)-y^2(v)| &= \sum_{u,v} M(u,v)|y(u)-y(v)|\cdot|y(u)+y(v)| \\
&\leq \sqrt{\sum_{u,v} M(u,v)(y(u)-y(v))^2}\sqrt{\sum_{u,v} M(u,v)(y(u)+y(v))^2} \\
&\leq \sqrt{\sum_{u,v} M(u,v)(y(u)-y(v))^2}\sqrt{\sum_{u,v} 2M(u,v)(y^2(u)+y^2(v))} \\
&= \sqrt{\sum_{u,v} M(u,v)(y(u)-y(v))^2}\sqrt{4dyy^T}.
\end{aligned}$$

□

The proof of the third claim is the main part of the argument.

PROOF:[Of Claim 4] Let $v_1,\ldots,v_n$ be an ordering of the vertices such that $y(v_1) \geq y(v_2) \geq y(v_n)$. Let t be the largest index such that $y(v_t) > 0$; recall that, by our assumptions, $t \leq n/2$.

We begin by removing the absolute value.

$$\sum_{u,v} M(u,v)|y^2(u) - y^2(v)| = 2\sum_{i=1}^{t}\sum_{j=i+1}^{n} M(v_i,v_j)(y^2(v_i) - y^2(v_j))$$

which we can rewrite as

$$= 2\sum_{k=1}^{t}\sum_{i\le k}\sum_{j>k} M(v_i,v_j)(y^2(v_k) - y^2(v_{k+1}))$$

because every edge (v_i, v_j), $i < j$, contributes to the second summation the correct value

$$\sum_{k=i}^{j-1} M(v_i,v_j)(y^2(v_k) - y^2(v_{k+1})) = M(v_i,v_j)(y^2(v_i) - y^2(v_j))$$

Let $S_k := \{v_1, \ldots, v_k\}$, then

$$\sum_{k=1}^{t}\sum_{i\le k}\sum_{j>k} M(v_i,v_j)(y^2(v_k) - y^2(v_{k+1})) = \sum_{k=1}^{t}(y^2(v_k) - y^2(v_{k+1}))\cdot\frac{1}{d}\cdot edges(S_k, V - S_k)$$

and, using the definition of expansion, we have the bound

$$\sum_{k=1}^{t}(y^2(v_k)-y^2(v_{k+1}))\cdot\frac{1}{d}\cdot edges(S_k, V-S_k) \ge \sum_{k=1}^{t} hk(y^2(v_k)-y^2(v_{k+1})) = h\sum_{k=1}^{t} y^2(v_k) = hyy^T$$

and so we have

$$\sum_{u,v} M(u,v)|y^2(u) - y^2(v)| \ge 2hyy^T$$

as required. □

Note that the proof in this section is algorithmic. Given an eigenvector $x_2 \perp (1, \ldots, 1)$ for λ_2 we can find a cut of expansion at most $\sqrt{2 - 2\lambda_2}$ by sorting the vertices of the graph as $v_1 \ldots, v_n$ so that $x_2(v_1) \ge \cdots \ge x_2(v_n)$ and then trying all cuts of the form $(\{v_1, \ldots, v_k\}, \{v_{k+1}, \ldots, v_n\})$.

4 References

The relationship between edge expansion and second eigenvalue in regular graphs was established by Alon [Alo86]. Sinclair and Jerrum prove similar inequalities in the more general setting of random walks on arbitrary undirected graphs [SJ89].

5 Exercises

1. Prove that if M is the transition matrix of a regular undirected graph G and $\lambda_1 \geq \cdots \lambda_n$ are its eigenvalues with multiplicities, then the number of eigenvalues equal to 1 is the same as the number of connected components of G.

   [If λ is an eigenvalue of M, then the set of vectors x such that $xM = \lambda x$ forms a linear space. For the solution of this problem you can assume the following result: the multiplicity of λ is the same as the dimension of linear space $\{x : xM = \lambda\}$.]

2. Let G be an undirected regular graph, M be its transition matrix, $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of M. Prove that $\lambda_n = -1$ if and only if G is bipartite.

3. Let G be an undirected regular graph, M be its transition matrix, $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of M. Prove that

$$\max_i |\lambda_i| = \max_{x \in \mathbb{R}^n, x \perp \mathbf{1}} \frac{||xM||}{||x||}$$

References

[Alo86] Noga Alon. Eigenvalues and expanders. *Combinatorica*, 6(2):83–96, 1986. 5

[SJ89] Alistair Sinclair and Mark Jerrum. Approximate counting, uniform generation and rapidly mixing Markov chains. *Information and Computation*, 82(1):93–133, 1989. 5