Notes for Lecture 10

Most of today's lecture was devoted to finishing the proof of Cheeger's inequality. These notes cover the additional discussion.

Tight Examples for Cheeger's Inequality

Let $G = (V, E)$ be a d-regular undirected graph with n vertices, M be its transition matrix, $\lambda_1 = 1 \geq \cdots \geq \lambda_n$ be the eigenvalues of M and $x_1, \ldots, x_n$ be a corresponding system of orthonormal eigenvectors, with $x_1 = \frac{1}{\sqrt{n}}(1, \ldots, 1)$.

Recall that we defined the normalized edge expansion of G as

$$h(G) := \min_{S \subseteq V,\ |S| \leq \frac{n}{2}} \frac{edges(S, V-S)}{d|S|}$$

and we proved Cheeger's inequality

$$2h \geq 1 - \lambda_2 \geq \frac{h^2}{2}$$

We will now show that there are graphs where $1 - \lambda_2$ is of the order of h, and graphs where it is order of h^2, thus showing that both sides of the inequality are essentially tight.

Before describing the examples (to spoil the surprise, they are the hypercube and the cycle, respectively), let us see what we should expect such examples to look like. We proved that

$$1 - \lambda_2 = \min_{x \in \mathbb{R}^n} \frac{\sum_{u,v} M(u,v)(x(u) - x(v))^2}{\frac{1}{n}\sum_{u,v}(x(u) - x(v))^2} \tag{1}$$

Furthermore, the eigenvector x_2 is a maximizer for the right-hand side.

We also proved that h is within a factor of 2 of the conductance Φ of G, which is defined as

$$\Phi(G) := \min_{S \subseteq V} \frac{edges(S, V-S)}{\frac{d}{n}|S| \cdot |V-S|} \tag{2}$$

and satisfies

$$\Phi(G) = \min_{x \in \{0,1\}^n} \frac{\sum_{u,v} M(u,v)(x(u) - x(v))^2}{\frac{1}{n}\sum_{u,v}(x(u) - x(v))^2} \tag{3}$$

So we have $\Phi(G) \geq 1 - \lambda_2$ simply because one quantity is a relaxation of the other, and $2h \geq 1 - \lambda_2$ follows because $h \geq \Phi/2$.

We can deduce that a graph for which the inequality is tight is a graph for which x_2 is (up to multiplication by a scalar and shift by an additive vector) a 0/1 vector.

The hypercube is such a graph. The t-dimensional hypercube H_t is a graph with $n = 2^t$ vertices, which we identify with $\{0,1\}^t$, and it is t-regular. Two vertices $u, v \in \{0,1\}^t$ are adjacent if and only if they differ in exactly one coordinate. In order to compute the eigenvalues of the hypercube, we are going to define 2^t orthogonal vectors x_a, one for each $a \in \{0,1\}^t$, and prove that they are all eigenvectors; the corresponding set of eigenvalues will then enumerate all the eigenvalues of H_t.

For $a \in \{0,1\}^t$, define the vector x_a as

$$x_a(u) = (-1)^{\sum_i a_i u_i}$$

Note that if $u, v \in \{0,1\}^t$ are two bit-vectors, and we denote by $u + v$ their bit-wise XOR, then $x_a(u + v) = x_a(u) \cdot x_a(v)$. Note also that $x_a(u) \cdot x_b(u) = x_{a+b}(u)$, where again $a + b$ denotes a bit-wise XOR. Finally, note that, for every $a \neq \mathbf{0}$,

$$\sum_u x_a(u) = 0$$

Let M be the transition matrix of H_t, so that $M(u, v) = \frac{1}{t}$ is u and v differ in exactly one coordinate, and $M(u, v) = 0$ otherwise.

Let $e_j \in \{0,1\}^t$ be the vector that is 1 in the j-th coordinate and 0 elsewhere. Then

$$\begin{aligned} (x_a M)(v) &= \sum_{u \text{ adjacent to } v} \frac{1}{t} x_a(u) \\ &= \sum_{j=1}^{t} \frac{1}{t} x_a(v + e_j) \\ &= \sum_{j=1}^{t} \frac{1}{t} x_a(v) \cdot x_a(e_j) \\ &= x_a(v) \cdot \sum_{j=1}^{t} \frac{1}{t} x_a(e_j) \end{aligned}$$

Which establishes that x_a is an eigenvector with eigenvalue $\frac{1}{t} \sum_j x_a(e_j)$; let us define $\lambda_a := \frac{1}{t} \sum_j x_a(e_j)$.

To see that the vectors are orthogonal, note that

$$x_a \cdot x_b^T = \sum_v x_a(v) x_b(v) = \sum_v x_{a+b}(v) = 0$$

Notice that the entries of the vectors x_a are ± 1, and so are essentially boolean values (the vector $\frac{1}{2} x_a + \frac{1}{2}\mathbf{1}$ is a 0/1 vector). Since one such vector is the maximizer in the right-hand size of (1), it follows that in the hypercube the conductance equals $1 - \lambda_2$. Let us explicitly compute conductance and eigenvalue gap.

Now, we can see that if a is a vector with ℓ non-zero entries then

$$\lambda_a = \frac{1}{t} \sum_j x_a(e_j) = 1 - 2\frac{\ell}{t}$$

So the unique largest eigenvalue is $\lambda_{\mathbf{0}} = 1$, and the second largest eigenvalue is $1 - \frac{2}{t}$.

Regarding the conductance, consider a dimension cut in the hypercube, that is a cut where S is, say, the set of vertices $u \in \{0,1\}^t$ whose first coordinate is 0. Then $|S| = |V - S| = n/2$, and the number of edges crossing the cut is also $|S| = n/2$. This proves that

$$\Phi \leq \frac{\frac{n}{2}}{\frac{t}{n} \cdot \frac{n}{2} \cdot \frac{n}{2}} = \frac{2}{t}$$

(The same cut, by the way, has expansion $\frac{1}{t}$, so the whole series of inequality $2h \geq \Phi \geq 1 - \lambda_2$ is proved tight by this example.)

Let us now turn to the proof that $1 - \lambda_2 \geq \frac{h^2}{2}$. We start from an eigenvector x_2 for λ_2 such that x_2 has at most $\frac{n}{2}$ positive entries, we define $y(v) := \max\{x(v), 0\}$ and we prove

$$2h \leq \frac{\sum_{u,v} M(u,v)|y^2(u) - y^2(v)|}{\sum_v y^2(v)} \leq \sqrt{4\frac{\sum_{u,v} M(u,v)|y(u) - y(v)|^2}{\sum_v y^2(v)}} \leq \sqrt{8(1 - \lambda_2)} \quad (4)$$

Consider again the case of the hypercube. Then (up to scaling) we can take x_2 to be the vector $x_{(1,0,\ldots,0)}$, that is, the vector such that $x_2(v) = (-1)^{v_1}$. The corresponding vector y will be such that $y(v) = 0$ if $v_1 = 1$ and $y(v) = 1$ if $v_1 = 0$. Doing the calculations, we see that

$$2h = \frac{\sum_{u,v} M(u,v)|y^2(u) - y^2(v)|}{\sum_v y^2(v)} = \frac{2}{t}$$

and

$$\sqrt{4\frac{\sum_{u,v} M(u,v)|y(u) - y(v)|^2}{\sum_v y^2(v)}} = \sqrt{8(1 - \lambda_2)} = \sqrt{\frac{16}{t}}$$

This means that all the loss is in the Cauchy-Schwarz step

$$\frac{\sum_{u,v} M(u,v)|y^2(u) - y^2(v)|}{\sum_v y^2(v)} \leq \sqrt{4\frac{\sum_{u,v} M(u,v)|y(u) - y(v)|^2}{\sum_v y^2(v)}}$$

Indeed, up to scaling, what happens is that we apply Cauchy-Schwarz to a vector that has tn entries, one for each pair (u, v) which is an edge, of value $|y^2(u) - y^2(v)|$, against another vector which is 1 everywhere. The former vector is 1 in a roughly $\frac{1}{t}$ fraction of entries, and zero elsewhere, hence the loss of a factor of the order $\frac{1}{\sqrt{t}}$. Note that, in contrast, Cauchy-Shwarz would have been tight if the values $|y^2(u) - y^2(v)|$ had been more or less the same for all edges (u, v).

The discussion so far suggests that, in a tight example for $1 - \lambda_2 \geq \frac{h^2}{2}$ we should expect an eigenvector x_2 for the second eigenvalue that

1. Has entries not concentrated around two values. Presumably, the entries are spread across a large number of possible values;

2. The values $|x^2(u) - x^2(v)|$, for (u, v) being an edge, are concentrated around a single value.

It turns out that the cycle has such properties, and is indeed a tight example for $1-\lambda_2 \geq \frac{h^2}{2}$.

It would be possible to explicitly characterize the eigenvalues and the eigenvectors of the cycle, the same way we did for the hypercube. There is, in fact, a general theory that characterizes eigenvalues and eigenvectors of all "Cayley graphs of abelian groups," of which the cycle and the hypercube are special cases. We will, instead, just prove the tightness (within constant factors) of the inequality.

Consider an (even) cycle with n vertices. Every cut is crossed by at least two edges, the degree is 2, and so the normalized edge expansion is

$$h = \min_{S:|S|\leq \frac{n}{2}} \frac{edges(S, V-S)}{d|S|} \geq \frac{2}{2 \cdot \frac{n}{2}} = \frac{2}{n}$$

Recall that a characterization of λ_2 is

$$1 - \lambda_2 = \min_{x: x \perp \mathbf{1}} \frac{\sum_{u,v} M(u,v)(x(u)-x(v))^2}{2\sum_v x^2(v)}$$

We define the following feasible solution x. If we number the vertices of the cycle as $1, \ldots, n$, then $x(i) = n/4 - i$ for $i = 1, \ldots, n/2$, and $x(i) = i - 3n/4$ for $i = n/2+1, \ldots, n$. In other words, the value of $x(i)$ goes from $n/4$ to $-n/4$ and then from $-n/4$ back to $n/4$ in increments of 1 when i goes from 1 to n.

For this choice of x, whose entries sum to zero so that $x \perp (1, \ldots, 1)$, we have

$$\sum_{u,v} M(u,v)(x(u)-x(v))^2 = n$$

and

$$\sum_u x(u)^2 = 4 \cdot \sum_{i=1}^{\frac{n}{4}} i^2 \approx 4 \cdot \frac{1}{3} \cdot \left(\frac{n}{4}\right)^3 = \frac{1}{48} n^3$$

so

$$1 - \lambda_2 \leq \frac{24}{n^2}$$

and $1 - \lambda_2 = \Theta(h^2)$.