Notes for Lecture 12

The Zig-Zag Graph Product

In this lecture we show that it is possible to "combine" a d-regular graph on D vertices and a D-regular graph on N vertices to obtain a d^2-regular graph on ND vertices which is a good expander if the two starting graphs are.

Let the two starting graphs be denoted by H and G respectively. Then, the resulting graph, called the *zig-zag product* of the two graphs is denoted by $G \textcircled{z} H$.

Using $\bar{\lambda}_2(G)$ to denote the eigenvalue with the second-largest absolute value for a graph G, we shall prove that $\bar{\lambda}_2(G \textcircled{z} H) \leq \bar{\lambda}_2(G) + \bar{\lambda}_2(H) + (\bar{\lambda}_2(H))^2$.

1 Replacement product of two graphs

We first describe a simpler product for a "small" d-regular graph on D vertices (denoted by H) and a "large" D-regular graph on n vertices (denoted by G). Assume that for each vertex of G, there is some ordering on its D neighbors. Then we construct the replacement product (Figure 1) $G \textcircled{r} H$ as follows:

- Replace each vertex of G with a copy of H (henceforth called a *cloud*). For $v \in V(G), j \in V(H)$, let (v, j) is the j-th vertex in the cloud of v.
- Let $(u, v) \in E(G)$ be such that v is the i-th neighbor of u and u is the j-th neighbor of v. Then $((u, i), (v, j)) \in E(G \textcircled{r} H)$. Also, if $(i, j) \in E(H)$, then $\forall v \in V(G)$ $((v, i), (v, j)) \in E(G \textcircled{r} H)$.

Note that the replacement product constructed as above has nD vertices and is $(d+1)$-regular.

2 Zig-zag product of two graphs

Given two graphs G and H as above, the zig-zag product $G \textcircled{z} H$ is constructed as follows (Figure 2):

- The vertex set $V(G \textcircled{z} H)$ is the same as in the case of the replacement product.
- $((u, i), (v, j)) \in E(G \textcircled{z} H)$ if there exist h and k such that $((u, i), (u, h))$, $((u, h), (v, k)))$ and $((v, k), (v, j))$ are in $E(G \textcircled{r} H)$ i.e. (v, j) can be reached from (u, i) by taking a step in the first cloud, then a step between the clouds and then a step in the second cloud (hence the name!).

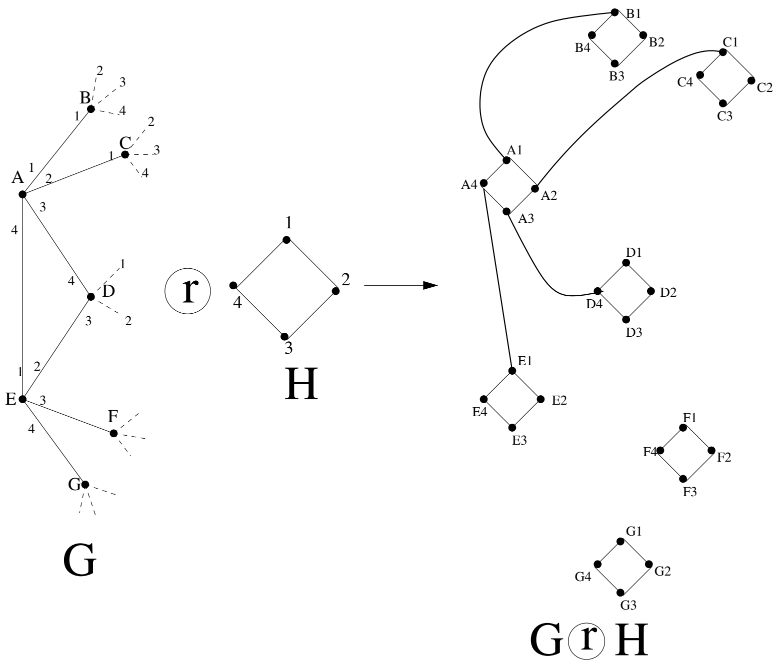

Figure 1: The replacement product of G and H (not all edges shown)

It is easy to see that the zig-zag product is a d^2-regular graph on nD vertices. Let $M \in \mathbb{R}^{([n]\times[D])\times([n]\times[D])}$ be the transition matrix of $G \textcircled{z} H$. Using the fact that each edge in $G \textcircled{z} H$ is made up of three steps in $G \textcircled{r} H$, we can write M as BAB, where

$$B[(u,i),(v,j)] = \begin{cases} 0 & \text{if } u \neq v \\ \frac{1}{d} \cdot \# \text{ edges between } i \text{ and } j \text{ in } H & \text{if } u = v \end{cases}$$

$$A[(u,i),(v,j)] = \begin{cases} 1 & \text{if } v \text{ is the } i\text{-th neighbor of } u \text{ and } u \text{ is the } j\text{-th neighbor of } v \\ 0 & \text{otherwise} \end{cases}$$

Here B is the adjacency matrix of the replacement product after deleting all the edges between clouds and A is the adjacency matrix containing *only* the edges between clouds. Note that A is the adjacency matrix for a matching and is hence a permutation matrix.

3 Eigenvalues of the zig-zag graph

Theorem 1 *If G is a D-regular graph on N vertices and H is a d-regular graph on D vertices, then*

$$\bar{\lambda}_2(G \textcircled{z} H) \leq \bar{\lambda}_2(G) + \bar{\lambda}_2(H) + (\bar{\lambda}_2(H))^2 \tag{1}$$

We know that

$$\bar{\lambda}_2(G) = \max_{x \perp \mathbf{1}, ||x||=1} \left| xMx^T \right| = \max_{x \perp \mathbf{1}} \frac{\left| xMx^T \right|}{xx^T}$$

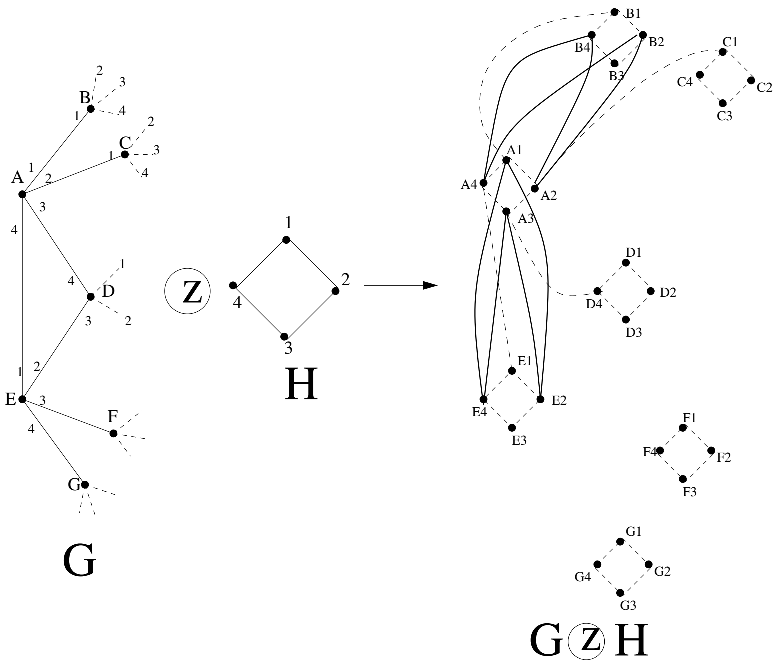

Figure 2: The zig-zag product of G and H and the underlying replacement product (not all edges shown)

Thus, it suffices to obtain a bound on the above expression for $G ⓩ H$ when G and H are good expanders. To provide an intuition for the proof consider two extreme cases for a cut in $G ⓩ H$. If the cut mostly includes or excludes entire clouds, then it can be viewed as a cut in G and the number of edges crossing it are almost the same as for the corresponding cut in G. If the cut splits almost all clouds in two parts, then one may think of it as N cuts in N copies of H. In both these cases then the number of edges crossing the cut will be "large" due the good expansion of G and H respectively. The following proof essentially breaks any vector x into the algebraic analogs of these two extremes.

PROOF: Given any vector $x \in \mathbb{R}^{ND}, x \perp \mathbf{1}$, one can write it as $x = x_{\|} + x_{\perp}$ where $x_{\|}$ is constant on each cloud and $x_{\perp}$, restricted to any cloud is perpendicular to $\mathbf{1}^D$ (the all 1's vector in D dimensions). In particular

$$\begin{aligned} x(u,i) &:= \frac{1}{D}\sum_j x(u,j) \\ x_{\perp}(u,i) &= x(u,i) - x_{\|}(u,i) \end{aligned}$$

We have

$$\begin{aligned} \left|xMx^T\right| = \left|xBABx^T\right| &= \left|(x_{\|}+x_{\perp})BAB(x_{\|}+x_{\perp})\right| \\ &\leq \left|x_{\|}BABx_{\|}^T\right| + 2\left|x_{\|}BABx_{\perp}^T\right| + \left|x_{\perp}BABx_{\perp}^T\right| \end{aligned}$$

We now analyze each of these terms separately.

$$\begin{aligned}
\left|x_\perp BABx_\perp^T\right| &= \left|x_\perp BA(x_\perp B)^T\right| \\
&\le ||x_\perp BA|| \cdot ||x_\perp B|| \quad (by\ Cauchy-Schwarz) \\
&= ||x_\perp B|| \cdot ||x_\perp B|| \quad (since\ A\ is\ a\ permutation\ matrix) \\
&\le \bar{\lambda}_2(H)\,||x_\perp|| \cdot \bar{\lambda}_2(H)\,||x_\perp|| \\
\Rightarrow \left|x_\perp BABx_\perp^T\right| &\le \bar{\lambda}_2(H)^2\,||x_\perp||^2 \qquad (2)
\end{aligned}$$

In the above $||x_\perp B|| \le \bar{\lambda}_2(H)\,||x||$ follows from the fact that the restriction of $x_\perp$ to any cloud is perpendicular to $\mathbf{1}^D$ and that B is a block-diagonal matrix whose action on the restriction is the same as that of the adjacency matrix of H. For the mixed term,

$$\begin{aligned}
\left|x_\perp BABx_\parallel^T\right| &= \left|x_\perp BA(x_\parallel B)^T\right| \\
&= \left|x_\perp BAx_\parallel^T\right| \qquad (\because\ x_\parallel\ is\ parallel\ to\ \mathbf{1}^D\ in\ each\ cloud) \\
&\le ||x_\perp B|| \cdot \left|\left|x_\parallel\right|\right| \\
&\le \bar{\lambda}_2(H) \cdot ||x_\perp|| \cdot ||x_\perp|| \\
&\le \frac{1}{2}\bar{\lambda}_2(H)(||x_\perp||^2 + ||x_\perp||^2) \quad (by\ Cauchy-Schwarz) \\
\Rightarrow \left|x_\perp BABx_\parallel^T\right| &\le \frac{1}{2}\bar{\lambda}_2(H)(\left|\left|x_\parallel\right|\right|^2 + ||x_\perp||^2) = \frac{1}{2}\bar{\lambda}_2(H)\,||x||^2 \qquad (3)
\end{aligned}$$

Let $y \in \mathbb{R}^N$ be the vector defined as $y(u) = \frac{1}{D}\sum_i x(u,i)$; note that $y(u) = x_\parallel(u,j)$ for all j, and so $||x_\parallel||^2 = D||y||^2$. Let C be the transition matrix of G. Then

$$\begin{aligned}
\left|x_\parallel BABx_\parallel^T\right| &= \left|x_\parallel Ax_\parallel^T\right| \\
&= \left|\sum_{u,i,v,j} x_\parallel(u,i)A(u,i),(v,j))x_\parallel(v,j)\right| \\
&= D\left|\sum_{u,v} y(u)C(u,v)y(v)\right| \\
&= D\left|yCy^T\right| \\
&\le D\bar{\lambda}_2(G)||y||^2 \\
&= \bar{\lambda}_2(G)\left|\left|x_\parallel\right|\right|^2 \\
\Rightarrow \left|x_\parallel BABx_\parallel^T\right| &\le \bar{\lambda}_2(G)\left|\left|x_\parallel\right|\right|^2 \qquad (4)
\end{aligned}$$

Note that $\left|yCy^T\right| \le \bar{\lambda}_2(G)||y||^2$ follows from the fact that $y\cdot\mathbf{1} = \sum_i y(i) = \frac{1}{D}\sum_i\sum_j x(v_{ij}) = 0$.

Combining the above bounds gives

$$\begin{aligned}
\left|xBABx^T\right| &\le \bar{\lambda}_2(G)\left|\left|x_\parallel\right|\right|^2 + \bar{\lambda}_2(H)^2\,||x_\perp|| + \bar{\lambda}_2(H)\,||x||^2 \\
\Rightarrow \left|xBABx^T\right| &\le (\bar{\lambda}_2(G) + \bar{\lambda}_2(H)^2 + \bar{\lambda}_2(H))\,||x||^2
\end{aligned}$$

Using the characterization of $\bar{\lambda}_2$, we have

$$\bar{\lambda}_2(G \textcircled{z} H) = \max_{x \perp \mathbf{1}, ||x||=1} \left| xBABx^T \right| \leq \bar{\lambda}_2(G) + \bar{\lambda}_2(H)^2 + \bar{\lambda}_2(H)$$

□

4 Using the Zig-Zag Product to Construct Expanders

First, we state without proof the existence of graphs with good expansion properties. The proof is simple and it will be given in a later lecture.

Theorem 2 *For every p prime and $t \leq p$ there is an explicit construction of a p^2 regular graph $G_{p,t}$ with p^{t+1} vertices such that $\bar{\lambda}_2(G_{p,t}) \leq \frac{t}{p}$.*

We will use the following special case of the previous theorem.

Corollary 3 *There is a constant d such that a d-regular graph H with d^4 vertices exists that satisfies $\bar{\lambda}_2(H) \leq \frac{1}{5}$.*

In particular, we can apply the theorem with $p = 37$ and $t = 7$, so that the degree is $(37)^2 = 1369$.

Using the H from above we shall construct inductively a family of progressively larger graphs, all of which are d^2-regular and have $\bar{\lambda}_2 \leq \frac{1}{2}$.

Let $G_0 := H^2$. For $k \geq 1$ let $G_{k+1} = (G_k^2) \textcircled{z} H$.

Theorem 4 *For each $k \geq 1$, G_k has degree d^2 and $\bar{\lambda}_2(G_k) \leq \frac{1}{2}$.*

Proof: We shall proceed by induction.

For the base case: $G_0 = H^2$ is d^2-regular and $\bar{\lambda}_2(H^2) = \frac{1}{25}$.

For the inductive step, assume the statement for k, *i.e.* G_k has degree d^2 and $\bar{\lambda}_2(G_k) \leq \frac{1}{2}$.

Then G_k^2 has degree $d^4 = |V(H)|$, so that the product $(G_k^2) \textcircled{z} H$ is defined. Moreover, $\bar{\lambda}_2(G_k^2) \leq \frac{1}{4}$. Applying the construction, we get that G_{k+1} has degree d^2 and

$$\bar{\lambda}_2(G_{k+1}) \leq \frac{1}{4} + \frac{1}{5} + \frac{1}{25} = \frac{49}{100} < \frac{1}{2}$$

This completes the proof. □

Finally note that G_k has $d^{4(k+1)}$ vertices.

5 References

The Zig-Zag graph product was defined and analysed by Reingold, Vadhan and Wigderson [RVW02]

References

[RVW02] Omer Reingold, Salil Vadhan, and Avi Wigderson. Entropy waves, the zig-zag graph product, and new constant-degree expanders. *Annals of Mathematics*, 155(1):157–187, 2002. 5