Notes for Lecture 13

Let G be a D-regular graph on N vertices and H be a d-regular graph on D vertices. We discussed how to define the graph $G \textcircled{z} H$ which is d^2-regular over ND vertices and we proved that $G \textcircled{z} H$ has good expansion if both G and H have good expansion. Today we give another analysis showing that as long as G and H have any noticeable expansion, it is possible to infer something non-trivial about the expansion of $G \textcircled{z} H$. We will use this analysis to show how one can "turn any graph into an expander."

1 Another Analysis of The Zig-Zag Graph Product

This is the main result of this lecture.

Theorem 1 (Main) *Suppose that* $\bar{\lambda}_2(G) \leq 1 - \epsilon_G$ *and* $\bar{\lambda}_2(H) \leq 1 - \epsilon_H$.
Then $\bar{\lambda}_2(G \textcircled{z} H) \leq 1 - \epsilon_G \epsilon_H^2$.

In order to prove the theorem, we need a result that shows that the transition matrix of a graph can always be seen as a convex combination of the transition matrix of a clique and of an "error" matrix. First, recall the definition of matrix norm.

Definition 1 (Matrix Norm) *Let* A *be a* $n \times m$ *matrix, then*

$$||A|| := \max_{x \in \mathbb{R}^n :\ ||x||=1} ||xA|| = \max_{x \in \mathbb{R}^n} \frac{||xA||}{||x||}$$

We have the following technical claim. (Recall that J_n is the $n \times n$ matrix that has a 1 in each entry, so that $\frac{1}{n} J_n$ is the transition matrix of a complete graph with self-loops. We omit the subscript when clear from the context.)

Lemma 2 *Let* G *be an undirected regular graph on* n *vertices,* M *be its transition matrix, and* $\lambda := \bar{\lambda}_2(G)$. *Then there is a matrix* E *such that* $||E|| \leq 1$ *and*

$$M = (1 - \lambda)\frac{1}{n} J + \lambda E$$

Proof: Let $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of M and $x_1, \ldots, x_n$ be a corresponding orthonormal set of eigenvectors, with $x_1 = \left(\frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}}\right)$. For a vector x, we can write $x = \alpha_1 x_1 + \cdots \alpha_n x_n$, where $\alpha_i = x \cdot x_i^T$, so that

$$xM = \alpha_1 x_1 + \lambda_2 \alpha_2 x_2 + \cdots + \lambda_n \alpha_n x_n$$

which we shall rewrite as

$$xM = (1 - \lambda)\alpha_1 x_1 + \lambda \cdot \left(\alpha_1 x_1 + \frac{\lambda_2}{\lambda}\alpha_2 x_2 + \cdots + \frac{\lambda_n}{\lambda}\alpha_n x_n\right)$$

Now, notice that $\alpha_1 x_1 = x \cdot x_1^T \cdot x_1 = x \cdot \frac{1}{n} J$.

Define

$$E := x_1^T \cdot x_1 + \frac{\lambda_2}{\lambda} x_2^T \cdot x_2 + \cdots + \frac{\lambda_n}{\lambda} x_n^T x_n$$

Then we have

$$xM = (1-\lambda) x \frac{1}{n} J + \lambda x E$$

and so

$$M = (1-\lambda) \frac{1}{n} J + \lambda E$$

It remains to bound $||E||$. Pick any unit vector x, and write it as $x = \alpha_1 x_1 + \cdots \alpha_n x_n$ then

$$\begin{aligned} ||xE|| &= ||\alpha_1 x_1 + \frac{\lambda_2}{\lambda}\alpha_2 x_2 + \cdots + \frac{\lambda_n}{\lambda}\alpha_n x_n|| \\ &= \sqrt{\alpha_1^2 + \left(\frac{\lambda_2}{\lambda}\alpha_2\right)^2 + \cdots \left(\frac{\lambda_n}{\lambda}\alpha_n\right)^2} \\ &\leq \sqrt{\alpha_1^2 + \cdots \alpha_n^2} = 1 \end{aligned}$$

□

We are now ready to give a proof of the Main Theorem. Consider the transition matrix of $G \textcircled{z} H$, and write it as BAB, where A is a permutation matrix and B is a block-diagonal matrix where each block is a copy of the transition matrix of H.

Use the Lemma we just proved to write the transition matrix of H as $\epsilon_H \frac{1}{D} J + (1-\epsilon_H) E_B$, for an error matrix such that $||E_B|| \leq 1$. Then we have

$$B = \epsilon_H U + (1-\epsilon_H) E$$

where U is a block-diagonal matrix that has $\frac{1}{D} J_D$ in each block (it would be the matrix B we would have if H had been a clique with self-loops), and E is a block-diagonal matrix that has a copy of E_B on each block. We'll leave it as an exercise to prove $||E|| \leq 1$.

We are ready to bound the $\bar{\lambda}_2$ parameter of BAB. Let x be a unit vector orthogonal to $(1, \cdots, 1)$.

$$\begin{aligned} |xBABx^T| &= |x(\epsilon_H U + (1-\epsilon_H)E) A (\epsilon_H U + (1-\epsilon_H)E) x^T| \\ &\leq \epsilon_H^2 |xUAUx^T| \\ &\quad + 2\epsilon_H (1-\epsilon_H) |xEAUx^T| \\ &\quad + (1-\epsilon_H)^2 |xEAEx^T| \end{aligned}$$

The main observation is that the vector xU is the same as the vector we called $x_{||}$ in the previous lecture, and so, by a calculation we have already done, we have

$$|xUAUx^T| \leq \bar{\lambda}_2(G) \cdot ||xU||^2 \leq 1 - \epsilon_G$$

For the other quantities, we can apply Cauchy-Schwarz and the bound on the matrix norm of E to derive

$$|xEAUx^T| \leq ||xEA|| \cdot ||xU|| \leq ||xE|| \cdot ||x|| \leq ||x||^2 = 1$$
$$|xEAEx^T| \leq ||xEA|| \cdot ||xE|| = ||xE||^2 \leq ||x||^2 = 1$$

and so

$$|xBABx^T| \leq \epsilon_H^2 - \epsilon_H^2\epsilon_G + 2\epsilon_H(1-\epsilon_H) + (1-\epsilon_H)^2 = 1 - \epsilon_H^2\epsilon_G$$

This concludes the proof of the Main Theorem.

There is also an alternative, equivalent, way of looking at this proof, that emphasizes the similarity with the proof we saw in the last lecture. In particular we can argue that if x is a vector orthogonal to $(1, \cdots, 1)$, then we can write

$$x = \epsilon_H x_{||} + (1-\epsilon_H)x'$$

where $x_{||}$ is as defined in the last lecture, and x' satisfies

$$||x'B|| \leq ||x||$$

(To prove tha above claim, write $x' = \frac{1}{1-\epsilon_H}x - \frac{\epsilon_H}{1-\epsilon_H}xU$, and notice that, since $B = \epsilon_H U + (1-\epsilon_H)E$, and $xUB = xU$, we have $x'B = xE$, and so $||x'B|| \leq ||xE|| \leq ||x||$.)

Then we can write

$$\begin{aligned} |xBABx^T| \quad \leq \quad & \epsilon_H^2 |x_{||}BABx_{||}^T| \\ & + 2\epsilon_H(1-\epsilon_H)|x_{||}BABx'^T| \\ & + (1-\epsilon_H)^2|x'BABx'^T| \end{aligned}$$

And, assuming $||x|| = 1$, we have the bounds

$$|x_{||}BABx_{||}^T| \leq 1 - \epsilon_G$$

$$|x_{||}BABx'^T| \leq 1$$

$$|x'BABx'^T| \leq 1$$

From which we recover $|xBABx^T| \leq 1 - \epsilon_H^2\epsilon_G$.

2 Turning Any Graph Into an Expander

In the previous lecture, given a good d-regular expander H on d^4 vertices, we defined a sequence of graphs $G_0, \ldots, G_k, \ldots$ by the recursion

$$\begin{aligned} G_0 &:= H^2 \\ G_{k+1} &:= G_k^2 \textcircled{z} H \end{aligned}$$

and proved that if the $\bar{\lambda}_2$ parameter of H is small enough that all graphs in the family have small $\bar{\lambda}_2$. By construction, G_k has d^{4+4k} vertices and is d^2-regular.

Now we want to show that, provided that H is a good expander, the construction eventually converges to a good expander *for every choice of a starting graph* G_0.

Theorem 3 *Let G be a d^2-regular graph, H be a d-regular graph on d^4 vertices such that $\bar{\lambda}_2(H) \leq \frac{1}{10}$. Define the sequence of graphs $G_0, \ldots, G_k, \ldots$ as*

$$\begin{aligned} G_0 &:= H^2 \\ G_{k+1} &:= G_k^2 \textcircled{z} H \end{aligned}$$

Then

$$\bar{\lambda}_2(G_k) \leq \max\left\{\frac{1}{2}, 1 - (1.2)^k \cdot (1 - \bar{\lambda}_2(G))\right\}$$

PROOF: The $k = 0$ base case is immediate; for the inductive step we need to prove that if we write $\bar{\lambda}_2(G_k) = 1 - \epsilon$ then

$$\bar{\lambda}_2(G_k^2 \textcircled{z} H) \leq \max\left\{\frac{1}{2}, 1 - (1.2)\epsilon\right\}$$

By the main theorem and the assumption on H we have

$$\bar{\lambda}_2(G_k^2 \textcircled{z} H) \leq 1 - (1 - (1-\epsilon)^2) \cdot .81$$

We consider two cases. If $\epsilon \geq \frac{1}{2}$, then $\bar{\lambda}_2(G_k^2) \leq \frac{1}{4}$, and

$$\bar{\lambda}_2(G_k^2 \textcircled{z} H) \leq 1 - \frac{3}{4} \cdot .81 < \frac{1}{2}$$

If $\epsilon \leq \frac{1}{2}$, then $(1 - (1-\epsilon)^2) = 2\epsilon - \epsilon^2 \geq 1.5\epsilon$, and

$$\bar{\lambda}_2(G_k^2 \textcircled{z} H) \leq 1 - 1.5 \cdot \epsilon \cdot .81 < 1 - 1.2 \cdot \epsilon$$

□

Suppose now that G is an arbitrary connected 3-regular graph, and define G_L to be the d^2-regular graph that is identical to G except that every vertex has $d^2 - 3$ self-loops. We can see (assuming $d^2 \geq 6$) that all the eigenvalues of G_L are non-negative, and

$$\bar{\lambda}_2(G_L) = \lambda_2(G_L) = \frac{d^2-3}{d^2} + \frac{3}{d^2}\lambda_2(G)$$

If G is a connected 3-regular graph, then $h(G) \geq \frac{2}{3n}$ and, by Cheeger's inequality, $\lambda_2(G) \leq 1 - \frac{2}{9n^2}$, so

$$\bar{\lambda}_2(G_L) \leq 1 - \frac{2}{3d^2n^2}$$

and, if we use G_L as a base case in the above theorem,

$$\bar{\lambda}_2(G_k) \leq \max\left\{\frac{1}{2}, 1 - (1.2)^k \cdot \frac{2}{3d^2n^2}\right\}$$

And for $k = O(\log n)$, we have $\bar{\lambda}_2(G_k) \leq \frac{1}{2}$. So it only takes a logarithmic number of steps to turn an arbitrary connected graph into a very good expander.

3 References

The analysis of the Zig-Zag product in Section 1 is due to Eyal Rozenman and Salil Vadhan [RV05].

References

[RV05] Eyal Rozenman and Salil Vadhan. Derandomized squaring of graphs. In *Proceedings of RANDOM'05*, pages 436–447. Springer-Verlag, 2005. 5