

Notes for Lecture 14

In the last lecture, we saw how a series of alternated squaring and zig-zag products can turn any connected graph into a good expander and, in particular, into a graph of diameter $O(\log n)$. Today we'll see how that construction naturally leads to Reingold's $O(\log n)$-space deterministic algorithm for ST-UCONN, the problem of connectivity in undirected graphs. There is an easy reduction from ST-UCONN in general graphs to the special case of 3-regular graphs, so we will only deal with the latter case.

We begin the lecture with the last missing piece from the expander constructions we have seen so far: how to construct a d-regular graph H with d^4 vertices and small $\bar{\lambda}_2$.

1 An Explicit Construction of Small Expanders

In this section we prove the following theorem, which was stated without proof in a past lecture.

Theorem 1 *For every prime p and integer $t < p$, there is a p^2-regular graph H with p^{t+1} vertices and $\bar{\lambda}_2(H) \leq \frac{t}{p}$.*

We let the vertex set of H be $V := \mathbb{F}_p^{t+1}$. For every $\alpha, \beta \in \mathbb{F}$, and every vertex $v = (v_0, \ldots, v_t)$, we have an edge connective v to the vertex $(v_0 + \beta, v_1 + \beta\alpha, \ldots, v_t + \beta\alpha^t)$. Note that the graph is p^2-regular as promised.

In order to analyse the eingenvalues of (the transition matrix of) H, we will present a set of $|V|$ orthogonal eigenvectors, and we will be guaranteed that their corresponding eigenvalues include all the eigenvalues of the graph. Although we know that there are real eigenvectors, we will present a system of eigenvectors with complex-valued entries.

We start with a few preliminary observations. Let $\omega := e^{2\pi i/p}$ be a primitive p-th root of unity. Observe that ω has the property that $\omega^p = 1$, and that

$$\omega^0 + \omega + \omega^2 + \cdots \omega^{p-1} = 0$$

where the above equality follows from the fact that, if we define $s := \sum_{j=0}^{p-1} \omega^j$, then we have $\omega \cdot s = s$, and so $s = 0$.

More generally, for every $1 \leq k \leq p - 1$, we have

$$\sum_{j=0}^{p-1} \omega^{kj} = 0$$

because the mapping $j \to kj \bmod p$ is a bijection, and so the above summation is just a reordering of the sum $\sum_{j=0}^{p-1} \omega^j$.

We now define our system of eigenvectors. For $b \in \mathbb{F}^{t+1}$, we define the vector x_b as

$$x_b(a) := \omega^{\sum_j a_j b_j}$$

We will need the following two "linearity" properties

$$x_b(u) \cdot x_b(v) = x_b(u+v) \tag{1}$$

and

$$x_a(v) \cdot x_b(v) = x_{a+b}(v) \tag{2}$$

Note also that $\overline{x_a(u)} = x_{-a}(u)$.

We can now prove that the vectors x_b are orthogonal. Take any two vectors x_a, x_b, $b \neq a$, and consider their inner product

$$\langle x_a, x_b \rangle = \sum_v x_a(v)\overline{x_b(v)} = \sum_v x_{a-b}(v)$$

$$= \sum_v \omega^{\sum_j (a_j - b_j)v_j} = \prod_j \sum_{v_j} \omega^{(a_j-b_j)v_j} = 0$$

where the last equality follows from the fact that if $a \neq b$ then there is an index j such that $a_j - b_j \not\equiv 0 \pmod p$, and so, for that j,

$$\sum_{v_j} \omega^{(a_j-b_j)v_j} = 0$$

Next, we prove that each of the vectors x_a is an eigenvector. If M is the transition matrix of H, then

$$(x_a M)(v) = \frac{1}{p^2} \sum_{\alpha,\beta} x_a(v - (\beta, \beta\alpha, \cdots \beta\alpha^t))$$

and, using "linearity,"

$$= \frac{1}{p^2} \sum_{\alpha,\beta} x_a(v) x_a(-(\beta, \beta\alpha, \cdots \beta\alpha^t))$$

$$= x_a(v) \cdot \frac{1}{p^2} \sum_{\alpha,\beta} x_a(\beta, \beta\alpha, \cdots \beta\alpha^t)$$

where we have eliminated the minus sign via the change of variable $\beta \to -\beta$.

Thus we established that each vector x_a is an eigenvector, with eigenvalue λ_a

$$\lambda_a := \frac{1}{p^2} \sum_{\alpha,\beta} x_a(\beta, \beta\alpha, \cdots, \beta\alpha^t)$$

When $a = (0, \ldots, 0)$, then $\lambda_{0,\ldots,0} = 1$, and the corresponding eigenvector is $(1, \ldots, 1)$, as usual in an undirected graph. We now bound all other eigenvalues in absolute value.

Let $a \neq (0, \ldots, 0)$, and define the polynomial $P_a(z) := a_0 + a_1 z + \cdots + a_t z^t$. This is a non-zero polynomial of degree t, and hence it has at most t roots.

$$
\begin{aligned}
|\lambda_a| &= \left| \frac{1}{p^2} \sum_{\alpha,\beta} x_a(\beta, \beta\alpha, \cdots, \beta\alpha^t) \right| \\
&= \left| \frac{1}{p^2} \sum_{\alpha,\beta} \omega^{a_0\beta + a_1\beta\alpha + \cdots + a_t\beta\alpha^t} \right| \\
&= \left| \frac{1}{p^2} \sum_{\alpha,\beta} \omega^{\beta \cdot P_a(\alpha)} \right| \\
&\leq \left| \frac{1}{p^2} \sum_{\alpha : P_a(\alpha)=0} \sum_{\beta} \omega^{\beta \cdot 0} \right| + \left| \frac{1}{p^2} \sum_{\alpha : P_a(\alpha)\neq 0} \sum_{\beta} \omega^{\beta \cdot P_a(\alpha)} \right| \\
&\leq \frac{t}{p}
\end{aligned}
$$

With the last inequality following from the fact that there are at most t values of α such that $P_a(\alpha) = 0$, and from the fact that for each α such that $P_a(\alpha) \neq 0$ we have, by previous calculations

$$
\sum_{\beta} \omega^{\beta \cdot P_a(\alpha)} = 0
$$

2 Reingold's Connectivity Algorithm

Let H be a d-regular graph on d^4 vertices such that $\bar{\lambda}_2(H) \leq \frac{1}{10}$. From the construction of the previous section, we can take $p = 71$ and $t = 7$, so that the graph is $d = (71)^2$-regular and has $(71)^8 = d^4$ vertices, while its $\bar{\lambda}_2$ parameter is at most $\frac{7}{71}$.

If G_0 is a d^2-regular graph on n vertices, and we define, for $k \geq 1$,

$$
G_k := G_{k-1}^2 \circledz H
$$

Then we proved in the previous lecture that each G_k is d^2-regular, it has $n \cdot d^{4k}$ vertices, and

$$
\bar{\lambda}_2(G_k) \leq \max\left\{ \frac{1}{2}, 1 - (1.2)^k (1 - \bar{\lambda}_2(G)) \right\}
$$

In particular, if G is a connected 3-regular graph, and G_0 is defined as G plus $d^2 - 3$ self-loops on each vertex, then

$$
\bar{\lambda}_2(G_k) \leq \max\left\{ \frac{1}{2}, 1 - (1.2)^k \cdot O\left(\frac{1}{n^2}\right) \right\}
$$

and, for $k = O(\log n)$, G_k has a $\bar{\lambda}_2$ parameter that is at most $1/2$. This also implies, by earlier calculations, that the diameter of G_k is at most $O(\log nd^{4k}) = O(\log n)$.

Let us give names to the vertices and edges of G_k.

Let us start from $G_1 := G_0^2 \textcircled{z} H$. G_0^2 has the same vertex set as G_0, but the (a,b)-th edge out of u in G_0^2 is the edge that connects u with v, provided that there is a vertex w in G_0 such that w is the a-th neighbor of v and u is the b-th neighbor of w. (In the above $a, b \in [d^2]$.)

In $G_0^2 \textcircled{z} H$, we have vertices of the type $(u,(a,b))$, and we have that the c-th neighbor of $(u,(a,b))$ is defined in the following way. Write $c = (c_1, c_2)$, so that c_1 and c_2 index edges in H. Let (f,g) be the c_1-th neihgbor of (a,b) in H. Let $(v,(h,m))$ be the unique neighbor of $(u,(f,g))$ outside the block of u in the replacement product. (That is, v is the (f,g)-th neighbor of u in G_0^2 and u is the (h,m)-th neighbor of v in G_0^2.) Finally, let (p,q) be the c_2-th neighbor of (f,g) in H.

Then $(v,(p,q))$ is the (c_1,c_2)-th neighbor of $(u,(a,b))$ in G_1.

In general, a vertex of G_k is of the form $(u, a_1, \ldots, a_k)$ where each a_i is an element of $[d^4]$.

Suppose now that G is not connected, but that a set of vertices S in G induces a connected component. As before, define G_0 to be G plus $d^2 - 3$ self-loops on each vertex, and define $G_k := G_{k-1}^2 \textcircled{z} H$. If we focus, at each step k, on the set of vertices of the form $(u, a_1, \ldots, a_k)$ with $u \in S$, then the graph they induce is the same we would have obtained as if we had started with G restricted to S and carried out the above procedure. In particular, when $k = O(\log n)$, this subgraph of G_k has logarithmic diameter. Note also that if S and T induce two distinct connected components in G, then vertices of the form $(u, a_1, \ldots, a_k)$ with $u \in S$ and $(v, b_1, \ldots, b_k)$ with $v \in T$ will form two distinct connected component in G_k.

These observations give the outline of Reingold's algorithm. Given a 3-regular graph G on n vertices and two vertices s, t, construct the graph G_k as above, with $k = O(\log n)$ chosen so that each connected component of G becomes a connected component of G_k having $\bar{\lambda}_2$ parameter at most $\frac{1}{2}$, and so diameter $O(\log n)$. Then, check whether $(s, a_1, \ldots, a_k)$ and $(t, a_1 \ldots, a_k)$ are connected, where the a_i can be chosen arbitrarily.

The second step can be implemented using $O(\log n)$ memory, because G_k has $n^{O(1)}$ vertices, $O(1)$ degree and $O(\log n)$ diameter. If the construction of G_k with $k = O(\log n)$ can be carried out by a $O(\log n)$ memory transducer, then the whole computation can be carried out using $O(\log n)$ space. (Recall that a log-space computation performed on the output of a log-space transducer can be performed in log-space.)

It remains to organize the recursive computation of the adjacency matrix of G_k in terms of G_{k-1} so that each step of the recursion can be implemented using only $O(\log n)$ global memory, shared between the various levels of recursion, and a constant number of extra bits of memory per level. We shall refer to Reingold's paper for the details.

3 References

Reingold's algorithm appeared in [Rei05].

References

[Rei05] Omer Reingold. Undirected ST-connectivity in log-space. In *Proceedings of the 37th ACM Symposium on Theory of Computing*, pages 376–385, 2005. 4