Notes for Lecture 17-18

In these two lectures we prove the first half of the PCP Theorem, the "Amplification Lemma," up to a statement about random walks on expanders that will be proved later.

Lemma 1 (Amplification) *For all Σ_0, there is a δ_0 such that $\forall c$, there exists Σ and a poly-time reduction R_1, mapping Max-2-CSP-Σ_0 to Max-2-CSP-Σ, such that the following hold for all instances $\mathcal{C}$ of Max-2-CSP-Σ_0*

1. *# of constraints of $R_1(\mathcal{C}) = O(\#$of constraints of $R_1(\mathcal{C}))$*
2. *$opt(\mathcal{C}) = 1 \Rightarrow opt(R_1(\mathcal{C})) = 1$*
3. *$opt(\mathcal{C}) \leq 1 - \delta \Rightarrow opt(R_1(\mathcal{C})) \leq 1 - c\delta$, provided $c\delta \leq \delta_0$*

By a result proved in the last lecture, without loss of generality we may assume the constraint graph G of $\mathcal{C}$ is a d-regular expander graph with $\bar{\lambda}_2(G) \leq \lambda < 1$, for an absolute constant λ indepedent of all the parameters of the lemma. Let $\mathcal{C}'$ be the weighted constraint satisfaction problem over $\Sigma = \Sigma_0^{1+d+d^2+\cdots+d^t}$ (for some t to be specified later) defined as follows:

Variables: For each variable v_i, $i = 1, \ldots, n$, of $\mathcal{C}$ we define a variable v_i' for $\mathcal{C}'$. Each variable v_i of $\mathcal{C}$ can be seen to assign an element of Σ_0 to its associated vertex v_i in G. The corresponding variable v_i' of $\mathcal{C}'$ can be interpreted to associate an element of Σ_0 to v_i and any vertex in G that is at most t edges away from v_i in G.

Constraints: First we define a distribution over paths in G. Then we will associate a constraint with each of these paths and will weight them by their probability. This will give us a weighted CSP.

Use the following randomized procedure to pick a path in G; this clearly defines a distribution over paths:

(D_1)

- Pick a starting vertex v_0 at random.
- For $i = 1$ to t: With probability 1/2 let $v_{i+1} \leftarrow v_i$. With probability 1/2 let $v_{i+1} \leftarrow$ a random neighbor of v_i.

Remark 1 *This can be viewed as a random walk of length t on the $2d$-regular graph G' which is the same as G except that G' has additionally d self loops at each vertex.*

The constraint associated with a path $v_0 \to v_1 \to \cdots \to v_t$ is satisfied in $\mathcal{C}'$ if and only if the variables corresponding to v_0 and v_t in $\mathcal{C}'$ give consistent assignments to all the vertices in the path and those assignments satisfy the constraint associated with each edge in the path.

Lemma 2 (Main) *The reduction described above satisfies the following:*

1. *$opt(\mathcal{C}) = 1 \Rightarrow opt(\mathcal{C}') = 1$*

2. $opt(\mathcal{C}) \leq 1-\delta \Rightarrow opt(\mathcal{C}') \leq 1 - \Omega_{\lambda,|\Sigma_0|}\left(\sqrt{t}\delta\right)$, *provided* $\sqrt{t}\delta < 1$

Proof: *1.* The first part is clear, because, if A is an assignment satisfying $\mathcal{C}$, then A assigns an element of Σ_0 to each vertex of the constraint graph G, so that the constraints associated with each edge are satisfied. We can use these same assignments to generate an assignment $\bar{A}$ for $\mathcal{C}'$ in the natural way. Then $\bar{A}$ will satisfy the constraint corresponding to every path $p : v_0 \to v_1 \to \cdots \to v_t$ of G because: 1) The values $\bar{A}(v_0)$ and $\bar{A}(v_t)$ are consistent by construction and 2) Every edge in the path will be satisfied by the values that $\bar{A}(v_0)$ and $\bar{A}(v_t)$ assign to its endpoints, because each edge of the path is either a self loop or an edge in G and A satisfies the constraint corresponding to every edge of G.

2. To show the second part, we will define a mapping from an assignment $\bar{A}$ for $\mathcal{C}'$ to an assignment A for $\mathcal{C}$ and we will show that

$$value[A(\mathcal{C})] \leq 1-\delta \Rightarrow value[\bar{A}(\mathcal{C}')] \leq 1 - \Omega_{\lambda,|\Sigma_0|} \tag{1}$$

This proves the theorem, because, if $opt(\mathcal{C}) \leq 1-\delta$, then, for any $\bar{A}$, the corresponding assignment A will satisfy at most a $1-\delta$ fraction of the constraints, and so, by (1), the value of $\bar{A}$ can be at most $1-\Omega(\sqrt{t}\delta)$. The mapping is defined as follows. Let us fix an assignment $\bar{A}$ for $\mathcal{C}'$. To define the assignment A for $\mathcal{C}$ to which $\bar{A}$ is mapped we use the following distribution over paths starting at any given vertex v_0 of G:

(D_2)

for $i = 1$ to $t/2$:
  With probability 1/2 let $v_{i+1} \leftarrow v_i$.
  With probability 1/2 let $v_{i+1} \leftarrow$ a random neighbor of v_i.

Let $A(v_0)$ be the value most likely to be given to v_0 by $\bar{A}(v_{t/2})$ according to this distribution. Notice that $\Pr[A(v_0) = \bar{A}(v_{t/2})] \geq \frac{1}{|\Sigma_0|}$.

We can pick a random constraint of $\mathcal{C}'$ using process (D_1). Equivalently, we can use the following randomized procedure for some fixed b, $b \in \{-t/2+1, \ldots, t/2\}$:

(D_3)

- pick a (directed) edge $(v_{t/2-b}, v_{t/2-b+1})$ from G' at random (recall that G' is a $2d$ regular graph resulting from G by adding d self loops to each vertex)
- choose $v_{t/2-b-1}, \ldots, v_0$ by taking a random walk in G' starting at $v_{t/2-b}$
- choose $v_{t/2-b+2}, \ldots, v_t$ by taking a random walk in G' starting at $v_{t/2-b+1}$

Intuition: Roughly speaking, if $b = 0$, the constraint picked by the above process will be violated with probability at least $\frac{1}{|\Sigma_0|^2}\frac{\delta}{2}$. The edge $(v_{t/2}, v_{t/2+1})$ picked at the first step of the procedure will be violated by A with probability at least $\frac{\delta}{2}$, because at least a δ fraction of the edges of G are not satisfied by A and with probability $\frac{1}{2}$ the first step of the above process will pick an edge of G (with probability $\frac{1}{2}$ it will pick one of the self loops added to G to get G'). Furthermore, as we commented above, $A(v_{t/2})$ will be consistent with $\bar{A}(v_0)$'s assignment for $v_{t/2}$ with probability at least $\frac{1}{|\Sigma_0|}$. Similarly, $A(v_{t/2+1})$ will be consistent with $\bar{A}(v_t)$'s assignment for $v_{t/2+1}$ with probability at least $\frac{1}{|\Sigma_0|}$ [1]. Note that this $\frac{1}{|\Sigma_0|^2}\frac{\delta}{2}$ bound is only a lower bound on the probability that the middle edge of a random constraint is contradicted. If a similar bound holds for $\Omega(\sqrt{t})$ edges in the middle of a random path and if the corresponding events are close to being disjoint, we would get our bound.

Formally: Let us fix an assignment $\bar{A}$ for $\mathcal{C}$ and the corresponding assignment A for $\mathcal{C}'$. The following definition is central to the remaining analysis.

Definition 1 *In a path $v_0 \to v_1 \to \cdots \to v_t$, a (directed) edge (v_i, v_{i+1}) is* faulty *if*

- *A contradicts constraint (v_i, v_{i+1}).*
- *$A(v_i)$ is consistent with $\bar{A}(v_0)$.*
- *$A(v_{i+1})$ is consistent with $\bar{A}(v_t)$.*

It is easy to see that, if a path contains a faulty edge, then the constraint corresponding to the path is violated by $\bar{A}$. Moreover, if $b \leq \sqrt{t}$, then the probability under (D_3) that the initially chosen edge is faulty should work out as before to be $\Omega\left(\frac{1}{|\Sigma_0|^2}\frac{\delta}{2}\right)$. So, if the corresponding events for different values of b were disjoint, we would be done. What we shall show next is that they are close enough to being disjoint.

Let F be the set of edges contradicted by A (or a subset of it of size $\delta|E|$ if the set of edges contradicted by A is bigger than that). For every directed edge $(u, v) \in F$, let us define the random variable $X_{(u,v),i}$ for a randomly chosen path $v_0 \to v_1 \to \cdots \to v_t$ in G':

$$X_{(u,v),i} = \begin{cases} 1 \quad \text{if} & \begin{array}{l} (v_i, v_{i+1}) = (u, v) \\ \bar{A}(v_0) \text{ is consistent with } A(v_i) \\ \bar{A}(v_t) \text{ is consistant with } A(v_{i+1}) \end{array} \\ \\ 0 & \text{otherwise} \end{cases}$$

Notice that $X_{(u,v),i}$ is the indicator variable that, when picking a random path of length t in G', the i-th vertex is u, the $(i+1)$-th vertex is v, $\bar{A}(v_0)$ is consistent with $A(v_i)$ and $\bar{A}(v_t)$ is consistent with $A(v_{i+1})$. If we define

$$N = \sum_{i=t/2-\sqrt{t}}^{t/2+\sqrt{t}} \left(\sum_{(u,v)\in F} X_{(u,v),i} \right),$$

then it is easy to see that the (weighted) fraction of constraints contradicted by $\bar{A}$ is at least $\Pr[N > 0]$. The following proposition completes the proof.

Proposition 3 $\Pr[N > 0] = \Omega(\sqrt{t} \cdot \delta)$

PROOF: We prove this in three parts:

[1] This is almost true because this is walk has length $\frac{t}{2} - 1$ instead of $\frac{t}{2}$ as is specified by our mapping and (D2).

1. $\Pr[N > 0] \geq \frac{\mathbb{E}[N]^2}{\mathbb{E}[N^2]}$
2. $\mathbb{E}[N] = \Omega(\sqrt{t} \cdot \delta)$
3. $\mathbb{E}[N^2] = O(\sqrt{t} \cdot \delta)$

(*Part 1*)

Claim 4 *For every non-negative random variable N:* $\Pr[N > 0] \geq \frac{(\mathbb{E}[N])^2}{\mathbb{E}[N^2]}$

PROOF: Let $\mathbf{1}_{N>0}$ be the indicator variable of the event $\{N > 0\}$, i.e.

$$\mathbf{1}_{N>0} = \begin{cases} 1, \text{if } N > 0 \\ 0, \text{otherwise} \end{cases}$$

From the Cauchy-Schwartz Inequality it follows that

$$\mathbb{E}[N] = \mathbb{E}[N \cdot \mathbf{1}_{N>0}] \leq \sqrt{\mathbb{E}[N^2]} \cdot \sqrt{\mathbb{E}[\mathbf{1}_{N>0}^2]} = \sqrt{\mathbb{E}[N^2]} \cdot \sqrt{\mathbb{E}[\mathbf{1}_{N>0}]} = \sqrt{\mathbb{E}[N^2]} \cdot \sqrt{\Pr[N > 0]}$$

The claim follows.□

(*Part 2*) It is enough to show the following.

Claim 5 *For every $(u, v) \in F$ and for every $i \in \{t/2 - \sqrt{t}, t/2 + \sqrt{t}\}$:* $\Pr[X_{(u,v),i} = 1] = \Omega[1/|E|]$.

PROOF:

$$\Pr[X_{(u,v),i} = 1] = \Pr[v_i = u \text{ and } v_{i+1} = v] \cdot \Pr\left[\begin{array}{c} \bar{A}(v_0) \text{ is consistent with } A(v_i) \\ \bar{A}(v_t) \text{ is consistant with } A(v_{i+1}) \end{array} \mid v_i = u \text{ and } v_{i+1} = v\right]$$

Note that $\Pr[v_i = u \text{ and } v_{i+1} = v] = \frac{1}{4|E|}$, because G' has $4|E|$ directed edges. To lower bound the other factor, we will need to compare the following experiments:

1. Pick a random directed edge (u, v) in G'. Do a random walk in G of length i from u, and a random walk of length $t - 1 - i$ from v. Call the end points of these two walks a and b.

2. Pick a random directed edge (u, v) in G'. Do a random walk in G' of length $t/2$ from u. Call the end point a. Do a random walk in G' of length $t/2$ from v. Call the end point b. Note that, by definition of the mapping from $\bar{A}$ to A, $\Pr[\bar{A}(a)$ is consistent with $A(u)] \geq 1/|\Sigma_0|$ and $\Pr[\bar{A}(b)$ is consistent with $A(v)] \geq 1/|\Sigma_0|$.

Once we condition on the number of steps that the walks took in G (which is G' without the self loops), the probability of consistency is the same in the above experiments. The rest of the analysis is the following:

In experiment 2:

$$
\begin{aligned}
\frac{1}{|\Sigma_0|} &\le \Pr[\bar{A}(a) \text{ consistent with } A(u)] = \\
&= \sum_l \Pr[\bar{A}(a) \text{ consistent with } A(u) | l \text{ steps taken in } G] \cdot \\
&\quad \cdot \Pr[l \text{ steps taken in } G \text{ when a total of } t/2 \text{ steps were taken in } G']
\end{aligned} \tag{S_2}
$$

While in experiment 1:

$$
\begin{aligned}
&\Pr[\bar{A}(a) \text{ consistent with } A(u)] = \\
&\quad = \sum_l \Pr[\bar{A}(a) \text{ consistent with } A(u) | l \text{ steps taken in } G] \cdot \\
&\qquad \cdot \Pr[l \text{ steps taken in } G \text{ when a total of } i \text{ steps were taken in } G']
\end{aligned} \tag{S_1}
$$

The range of l in the above summations is not the same when $i \neq \frac{t}{2}$. Ignoring this fact, since the first factor of every term in the above summations is the same, to finish the claim it would be enough to show something of the flavor

$$
\begin{aligned}
&\Pr[l \text{ steps taken in } G \text{ when a total of } i \text{ steps were taken in } G'] \\
&\quad = \Omega(\Pr[l \text{ steps taken in } G \text{ when a total of } t/2 \text{ steps were taken in } G'])
\end{aligned}
$$

This is not always true, but it is true when the value of i is close to $\frac{t}{2}$ [2] and the value of l is close to its expectation, that is, around $t/4$. So, let us remove the tails from the above summations!

It is not hard to see that there exists some $c = c(\log |\Sigma_0|)$ such that (S_2) implies

$$
\begin{aligned}
\frac{1}{2|\Sigma_0|} &\le \sum_{l=\frac{t}{4}-c\sqrt{t}}^{\frac{t}{4}+c\sqrt{t}} \Pr[\bar{A}(a) \text{ consistent with } A(u) | l \text{ steps taken in } G] \cdot \\
&\quad \cdot \Pr[l \text{ steps taken in } G \text{ when a total of } t/2 \text{ steps were taken in } G'] \\
&\le \sum_{l=\frac{t}{4}-c\sqrt{t}}^{\frac{t}{4}+c\sqrt{t}} \Pr[\bar{A}(a) \text{ consistent with } A(u) | l \text{ steps taken in } G] \cdot \\
&\quad \cdot O\left(\Pr[l \text{ steps taken in } G \text{ when a total of } i \text{ steps were taken in } G']\right) \\
&\le O(\Pr[\bar{A}(a) \text{ consistent with } A(u) \text{ in first experiment}])
\end{aligned}
$$

The second inequality in the above derivation relies on the fact that, if we flip i fair coins $i \in \{\frac{t}{2} - \sqrt{t}, \dots, \frac{t}{2} + \sqrt{t}\}$, then the probability of getting l heads where $l \in \{\frac{t}{4} -$

[2] Recall that in the statement of the claim we asked $i \in \{t/2 - \sqrt{t}, t/2 + \sqrt{t}\}$.

$c\sqrt{t}, \ldots, \frac{t}{4} + c\sqrt{t}\}$ is $\Theta(\frac{1}{\sqrt{i}}) = \Theta(\frac{1}{\sqrt{t}})$. The third inequality results from removing the tails from summation (S_1). □

Putting everything together we get

$$\mathbb{E}\left[\sum_{i=\frac{t}{2}-\sqrt{t}}^{\frac{t}{2}+\sqrt{t}} X_{(u,v),i}\right] \geq 2\sqrt{t} \cdot \frac{1}{4|E|} \cdot \Omega\left(\frac{1}{|\Sigma_0|}\right) = \Omega_{|\Sigma_0|}(\sqrt{t}) \cdot \frac{1}{4|E|}.$$

And so

$$\mathbb{E}[N] = \Omega_{|\Sigma_0|}\left(\sqrt{t}\right) \cdot \delta|E| \cdot \frac{1}{4|E|} = \Omega_{|\Sigma_0|}(\sqrt{t}\delta).$$

(*Part 3*)

Claim 6 $\mathbb{E}\left[N^2\right] = O_\lambda(\sqrt{t}\delta)$.

PROOF: For a path $v_0 \to v_1 \to \ldots \to v_t$, randomly chosen as described above, we define random variables $X'_{(u,v),i}$ and N' as follows

$$X'_{(u,v),i} = \begin{cases} 1, & if\ v_i = u\ and\ v_{i+1} = v \\ 0, & otherwise \end{cases}$$

$$N' = \sum_{i=\frac{t}{2}-\sqrt{t}}^{\frac{t}{2}+\sqrt{t}} \sum_{(u,v)\in F} X'_{(u,v),i}$$

Clearly, $N' \geq N$ and, thus, it suffices to bound $\mathbb{E}\left[N'^{\,2}\right]$. The bound follows from the following result about random walks in expanders, which we will prove in the next lecture, applied to the $\ell = 2\sqrt{t}$ steps of the random walk between step $\frac{t}{2} - \sqrt{t}$ and step $\frac{t}{2} + \sqrt{t}$.

Lemma 7 *Let $G = (V, E)$ be a $d-$regular graph with $\bar{\lambda}_2(G) \leq \lambda < 1$, let $F \subseteq E$, and define $\delta = |F|/|E|$. Pick a random walk of length ℓ in G, and let M be the number of edges of F traversed in the walk.*

Then

$$\mathbb{E}[M^2] = O_\lambda(\delta\ell + \delta^2\ell^2)$$

The lemma above gives us a bound $\mathbb{E}[N'^2] = O_\lambda(\delta\sqrt{t} + \delta^2 t)$, but recalling that $\delta\sqrt{t} < 1$, we have $\mathbb{E}[N'^2] = O_\lambda(\delta\sqrt{t})$ as required.

(end of proof of claim 6) □

(end of proof of proposition 3) □

(end of proof of lemma 2) □