Notes for Lecture 19

In this lecture we prove the only remaining missing step in the proof of the first half of the PCP Theorem, and we begin a description of the second half of the proof.

1 A Lemma on Random Walks on Expanders

In the last lecture, we stated the following result without proof.

Lemma 1 *Let $G = (V, E)$ be a d-regular graph with $\bar{\lambda}_2(G) \leq \lambda < 1$, let $F \subseteq E$, and define $\delta = |F|/|E|$. Pick a random walk of length ℓ in G, and let M be the number of edges of F traversed in the walk.*

Then

$$\mathbb{E}[M^2] = O_\lambda(\delta\ell + \delta^2\ell^2)$$

We denote the random walk as the sequence $v_0, v_1, \ldots, v_\ell$ (where each v_i is a random variable), and we introduce the 0/1 random variables $X_1, \ldots, X_\ell$, defined so that $X_i = 1$ if $(v_{i-1}, v_i) \in F$, and $X_i = 0$ otherwise. Hence

$$M = X_1 + X_2 + \cdots X_\ell$$

and, using linearity of expectation and the fact that $X_i^2 = X_i$,

$$\mathbb{E}[M^2] = \sum_{i,j} \mathbb{E}[X_i X_j] = \sum_i \mathbb{E}[X_i] + 2\sum_i \sum_{j>i} \mathbb{E}[X_i X_j]$$

Every edge in a random walk is uniformly distributed, and so

$$\sum_i \mathbb{E}[X_i] = \delta\ell$$

It remains to bound the cross products. Our strategy will be to show that, for every i and $j > 1$, we have

$$\mathbb{E}[X_i X_j] \leq \delta^2 + \delta\lambda^{j-i-1} \tag{1}$$

Assuming that we have Equation (1), then for every i we have

$$\sum_{j>i} \mathbb{E}[X_i X_j] \leq \delta^2\ell + \delta \sum_{k=0}^{j-i-1} \lambda^k \leq \delta^2\ell + \delta\frac{1}{1-\lambda}$$

and so

$$2\sum_i \sum_{j>i} \mathbb{E}[X_i X_j] = O_\lambda(\delta^2\ell^2 + \delta\ell)$$

as required.

We now turn to the proof of Equation (1). First, note that

$$
\begin{aligned}
\mathbb{E}[X_i X_j] &= \Pr[(v_{i-1}, v_i) \in F \wedge (v_{j-1}, v_j) \in F] \\
&= \Pr[(v_{i-1}, v_i) \in F] \cdot \Pr[(v_{j-1}, v_j) \in F \mid (v_{i-1}, v_i) \in F] \\
&= \delta \cdot \Pr[(v_{j-1}, v_j) \in F \mid (v_{i-1}, v_i) \in F]
\end{aligned}
$$

Which means that proving Equation (1) reduces to proving the bound

$$
\Pr[(v_{j-1}, v_j) \in F \mid (v_{i-1}, v_i) \in F] \leq \delta + \lambda^{j-i-1} \tag{2}
$$

Now, the distribution of the edge (v_{j-1}, v_j) on a random walk conditioned on (v_{i-1}, v_i) is the same as the distribution of the edge (u_{j-i-1}, u_{j-i}) in a random walk $u_0, \ldots, u_{j-i}$ where u_0 is chosen to be a random endpoint of a random edge of F, and the subsequent steps are a length-$(j-i)$ random walk in G.

Equation (2) can be abstracted as the following claim.

Lemma 2 *Let $G = (V, E)$ be a $d-$regular graph with $\bar{\lambda}_2(G) \leq \lambda < 1$ and let $F \subseteq E$. Let $u_0, \ldots, u_k$ be a random walk in G where the starting point u_0 is chosen by picking a random edge in F and then a random endpoint of the edge.*

Then the probability that (u_{k-1}, u_k) is in F is at most

$$
\frac{|F|}{|E|} + \lambda^{k-1}
$$

PROOF: Let M be the transition matrix of a random walk on the graph and let $d_F(v)$ denote the number of edges incident on v that belong to F. Then the initial distribution vector x (describing the distribution of u_0) is of the form $x(v) = \frac{d_F(v)}{2|F|}$. The distribution z after $k-1$ steps (the distribution of u_{k-1}) is given by $z = xP^{k-1}$. If the walk is at vertex v after $k-1$ steps, then the probability that the last step will be along an edge in F is $\frac{d_F(v)}{d} = \frac{2|F|}{d} x(v)$. Thus

$$
\Pr\left[(u_{k-1}, u_k) \in F\right] \;=\; \sum_v z(v) \frac{d_F(v)}{d} \;=\; \frac{2\,|F|}{d} z x^T \;=\; \frac{2\,|F|}{d} x M^{k-1} x^T
$$

To obtain a bound on $xM^{k-1}x^T$, we split x as $x = x_{\|} + x_{\perp}$ where $x_{\|}$ and $x_{\perp}$ are respectively parallel and perpendicular to the uniform distribution. Specifically, $x_{\|}(v) = \frac{1}{n}$ and $x_{\perp}(v) = x(v) - \frac{1}{n}$. Then

$$
\begin{aligned}
xM^{k-1}x^T &= x_{\|} M^{k-1} x^T + x_{\perp} M^{k-1} x^T \\
&\leq \langle x_{\|}, x \rangle + \left|\left| x_{\perp} M^{k-1} \right|\right| \, ||x|| && \text{(because } x_{\|} M = x_{\|}\text{)} \\
&\leq \frac{1}{n} + \lambda^{k-1} \, ||x||^2 && \text{(since } ||x_{\perp} M|| \leq \lambda \, ||x_{\perp}|| \text{ and } ||x_{\perp}|| \leq ||x||\text{)}
\end{aligned}
$$

Also

$$
\begin{aligned}
||x||^2 &= \sum_v \frac{(d_F(v))^2}{(2|F|)^2} \leq \max_v \frac{d_F(v)}{2|F|} \sum_v \frac{d_F(v)}{2|F|} \\
&\leq \frac{d}{2|F|} \qquad (\text{since } \sum_v d_F(v) = 2|F| \text{ and } d_F(v) \leq d)
\end{aligned}
$$

Using these, we obtain the required result as

$$
\begin{aligned}
\Pr[(u_{k-1}, u_k) \in F] &= \frac{2|F|}{d} x M^{k-1} x^T \\
&\leq \frac{2|F|}{d} \left[\frac{1}{n} + \lambda^{k-1} \frac{d}{2|F|} \right] \\
&= \frac{2|F|}{dn} + \lambda^{k-1} \\
&= \frac{|F|}{|E|} + \lambda^{k-1}.
\end{aligned}
$$

□

2 An Overview of the Rest of the Proof of the PCP Theorem

To complete the proof of the PCP Theorem we will need to establish the following result.

Lemma 3 (Range Reduction) *$\exists \Sigma_0$, $\exists c_0 > 0$, such that for all Σ, there exists a poly-time R_2, mapping Max-2-CSP-Σ to Max-2-CSP-Σ_0 such that:*

- *For every C, size$(R_2(C)) = O($size(C);*
- *If C is satisfiable, then $R_2(C)$ is satisfiable;*
- *If opt$(C) \leq 1 - \delta$, then opt$(R_2(C)) \leq 1 - c_0\delta$.*

We say that an instance of Max-2-CSP-Σ is in "projection form" if every constraint is of the form $x = f(y)$, where $f : \Sigma \to \Sigma$ can be arbitrary function, possibly dependent on the constraint.

The main result in the proof of Lemma 3 will be the following.

Lemma 4 (Reduction to Boolean CSP) *There is a q and a c_1 such that for all Σ, there exists a poly-time reduction R_B, mapping Max-2-CSP-Σ to Max q-CSP-$\{0,1\}$. such that:*

- *For every C, size$(R_B(C)) = O($size(C);*
- *If C is satisfiable, then $R_B(C)$ is satisfiable;*
- *If opt$(C) \leq 1 - \delta$, then opt$(R_B(C)) \leq 1 - c_1\delta$.*

The proof of Lemma 3 is completed by the following easier reduction.

Lemma 5 (Reduction to Projection Form) *For all Σ and q, there exists a poly-time reduction R_P, mapping Max q-CSP-Σ to Max-2-CSP-Σ^q in projection form such that:*

- *For every C, size$(R_P(C)) = O($size(C);*
- *If C is satisfiable, then $R_P(C)$ is satisfiable;*
- *If opt$(C) \leq 1 - \delta$, then opt$(R_P(C)) \leq 1 - \delta/q$.*

To prove Lemma 3, we start from an instance C of Max-2-CSP-Σ and we use Lemma 5 to reduce it to an instance C_1 of Max-2-CSP-Σ^2 in projection form. Then we use Lemma 4 to reduce C_1 to an instance C_2 of Max q-CSP-$\{0,1\}$. Finally, we use Lemma 5 to reduce C_2 to an instance C_3 of Max 2-CSP-$\{0,1\}^q$. This proves Lemma 3 with $\Sigma_0 = \{0,1\}^q$, where q is the constant of Lemma 4, and with $c_0 = c_1/2q$, where c_1 is the constant of Lemma 4.

We conclude this lecture with a proof of Lemma 5.

Proof: [Of Lemma 5] Let C be an instance of Max q-CSP-Σ with variables $x_1, \ldots, x_n$ and constraints $C_1, \ldots, C_m$. The reduction produces a new instance that has variables $x_1, \ldots, x_n, y_1, \ldots, y_m$, that is, the same set of original variables, plus an extra variable per constraint of C. We also fix a surjective mapping of $\Sigma^q \to \Sigma$ so that we may think of an assignment to the x_i in the new instance as an assignment to the x_i in the original instance.

Each constraint C_j, over variables $x_{j_1}, \ldots, x_{j_q}$, is mapped into q new constraints. The i-th such constraint, over variables x_{j_i} and y_j, requires that, if we think of the value of y_j as specifying an assingment to $x_{j_1}, \ldots, x_{j_q}$, then such assignment must satisfy C_j and must be consistent with the assigned value of x_{j_i}.

Now we see that if opt$(C) \leq 1 - \delta$, then any assignment to the x_j contradicts at least δm constraints of C, and that, for each such constraint, no matter what is the assignment to the y_j, at least one of the q constraints derived from it will be contradicted in the new instance. Hence, every assignment to the new instance contradicts at least δm of the qm constraints, and the optimum is at most $1 - \delta/q$. $\Box$