Midterm

Due Tuesday, April 15, by email or in class. Work on the exam individually.

1. **A Hierarchy Theorem for Circuits [25/100].** Prove that for all sufficiently large n there is a function $f : \{0,1\}^n \to \{0,1\}$ computable by a circuit of size n^3 but not by any circuit of size $\leq n^3/(100 \log n)$.

   [There are several possible arguments to prove this "hierarchy theorem for circuits." Some arguments give stronger bounds: you can get f to not have circuits of size $n^3 - O(n)$, or even $n^3 - O(1)$.]

2. **Circuit Lower Bounds in the Polynomial Hierarchy [25/100].**

   (a) **[15/100]** Prove that there is a language in the Polynomial Hierarchy (you should be able to get it in Σ_4) that is not solvable by any family of circuits of size $O(n^3)$.

   (b) **[10/100]** Using the previous result and the Karp-Lipton theorem, prove that there is a language in Σ_2 that is not solvable by any family of circuits of size $O(n^3)$.

3. **Log-Space Counting Problems [25/100].**

   Consider the following two definitons of log-space counting problems.

   A function $f : \{0,1\}^* \to \mathbb{N}$ is in $\#L1$ if there is a non-deterministic Turing machine M_f that on input x of length n uses $O(\log n)$ space and is such that the number of accepting paths of $M_f(x)$ equals $f(x)$.

   A function $f : \{0,1\}^* \to \mathbb{N}$ is in $\#L2$ is there is a relation $R(\cdot,\cdot)$ that is decidable in log-space and a polynomial p such that if $R(x,y)$ then $|y| \leq p(|x|)$ and such that $f(x)$ equals $|\{y : R(x,y)\}|$.

   Prove that all functions in $\#L1$ can be computed in polynomial time (15/100), while $\#L2$ equals $\#P$ (10/100).

4. **Powering and Edge Expansion [25/100].** Recall that if G is a d-regular graph with transition matrix M, then G^k is the d^k-regular graph with transition matrix M^k that has one edge for each path of length k in G (with repetitions).

   (a) **[20/100]** Prove that if $\overline{h}(G) \geq \epsilon$, then there is a $k = k(\epsilon)$ that depends only on ϵ and not on the size of G such that $\overline{h}(G^k) \geq \frac{1}{10}$.

   (b) **[5/100]** Provide a counterexample to the following statement:

   $$\overline{h}(G^2) \geq \min\left\{\frac{1}{10}\ ,\ 1.01 \cdot \overline{h}(G)\right\}$$

   [Note: the statement may be true (it's an open question) if G^2 is replaced by G^3. If you can prove it for G^k, for some constant k, it can be your project.]