Practice Midterm

1. A directed graph $G = (V, E)$ is *strongly connected* if for any two vertices $u, v \in V$ there is a directed path in G from u to v. Let strong-CONN be the problem of deciding whether a given graph is strongly connected.

   (a) Show that strong-CONN is in **NL**.

   (b) Prove the **NL**-completeness of strong-CONN by giving a log-space reduction from ST-CONN to strong-CONN.

2. Suppose that there is a deterministic polynomial-time algorithm A that on input (the description of) a circuit C produces a number $A(C)$ such that

$$\mathbf{Pr}_x[C(x) = 1] - \frac{2}{5} \leq A(C) \leq \mathbf{Pr}_x[C(x) = 1] + \frac{2}{5} .$$

   (a) Prove that it follows $\mathbf{P} = \mathbf{BPP}$.

   (b) Prove that there exists a deterministic algorithm A' that, on input a circuit C and a parameter ϵ, runs in time polynomial in the size of C and in $1/\epsilon$ and produces a value $A'(C, \epsilon)$ such that

$$\mathbf{Pr}_x[C(x) = 1] - \epsilon \leq A'(C, \epsilon) \leq \mathbf{Pr}_x[C(x) = 1] + \epsilon .$$

   (c) Prove that there exists a deterministic algorithm A'' that, on input a circuit C computing a function $f : \{0, 1\}^n \to \{1, \ldots, k\}$ and a parameter ϵ, runs in time polynomial in the size of C, in $1/\epsilon$ and in k, and produces a value $A''(C, \epsilon)$ such that

$$\mathbf{E}_x[f(x)] - \epsilon \leq A''(C, \epsilon) \leq \mathbf{E}_x[f(x)] + \epsilon .$$

   [For this question, you can think of C as being a circuit with $\log k$ outputs, and the outputs of $C(x)$ are the binary representation of $f(x)$.]

3. Prove that if $PSPACE \subseteq SIZE(\text{poly}(n))$, then $PSPACE$ is contained in the polynomial hierarchy.

   [Note: this may be harder than the other problems]

4. Consider a two-dimensional grid graph , that is, the graph whose set of vertices is $\{1, \ldots, \sqrt{n}\} \times \{1, \ldots, \sqrt{n}\}$ and such that a vertex (i, j) has the four neighbors

$$(i + 1 \bmod \sqrt{n}, j), (i - 1 \bmod \sqrt{n}, j), (i, j + 1 \bmod \sqrt{n}), (i, j - 1 \bmod \sqrt{n})$$

   Let M be the transition matrix of this graph and let $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of M. Prove that $\lambda_2 \leq 1 - \Omega(1/n)$.

   [Hint: recall how we prove that in an n-cycle $\lambda_2 \leq 1 - \Omega(1/n^2)$.]

5. A directed graph is d-regular if every vertex has in-degree d and out-degree d. Prove that the $ST-CONN$ problem in *directed regular graphs* is in $\mathbf{L}$.

   [Hint: give a reduction to the undirected case.]