# 1 Matchings in graphs

This week we will be talking about finding matchings in graphs: a set of edges that do not share endpoints.

**Definition 1.1** (Maximum Matching)**.** *Given an undirected graph $G = (V, E)$, find a subset of edges $M \subseteq E$ of maximum size such that every pair of edges $e, e' \in M$ do not share endpoints $e \cap e' = \emptyset$.*

**Definition 1.2** (Perfect Matching)**.** *Given an undirected graph $G = (V, E)$ where $|V| = n$ is even, find a subset of edges $M \subseteq E$ of size $n/2$ such that every pair of edges $e, e' \in M$ do not share endpoints $e \cap e' = \emptyset$. That is, every node must be covered by the matching $M$.*

Obviously, any algorithm for Maximum Matching gives an algorithm for Perfect Matching. It is an exercise to show that if one can solve Perfect Matching in $T(n)$ time, then one can solve Maximum Matching in time $\tilde{O}(T(2n))$. The idea is to binary search for the maximum $k$ for which there is a matching $M$ with $|M| \geq k$. To check whether such $M$ exists, we can add a clique on $n - 2k$ nodes to the graph and connect it to the original graph with all possible edges. The new graph will have a perfect matching if and only if the original graph had a matching with $k$ edges.

We will focus on Perfect Matching and give algebraic algorithms for it. Because of the above reduction, this will also imply algorithms for Maximum Matching. The idea will be to define some matrix such that the determinant of this matrix is non-zero if and only if the graph has a perfect matching.

## 1.1 The Tutte Matrix

**Definition 1.3.** *For a graph $G = (V, E)$ with $|V| = n$, the following $n \times n$ matrix $T$ is the Tutte matrix of $G$:*

$$T[i,j] = \begin{cases} 0 & \textit{if } i = j \textit{ or if } (i,j) \notin E \\ x_{i,j} & \textit{if } (i,j) \in E \textit{ and } i < j \\ -x_{i,j} & \textit{if } (i,j) \in E \textit{ and } i > j \end{cases}$$

The following theorem is at the core of all the algorithms for Perfect Matching that we will discuss.

**Theorem 1.1** (Tutte)**.** *For any graph $G = (V, E)$, the determinant of the Tutte matrix $T$ is non-zero if and only if $G$ contains a perfect matching.*

$$\det(T) \neq 0 \iff G \textit{contains a perfect matching.}$$

*Proof.* By the definition of the determinant:

$$\det(T) = \sum_{\sigma \in S_n} (-1)^{sign(\sigma)} \cdot \underbrace{\prod_{i=1}^{n} T[i, \sigma(i)]}_{f_\sigma} \tag{1}$$

where $S_n$ is the set of permutations of $[n]$ and $sign(\sigma)$ is the parity of inversions for $\sigma$, i.e. the number of pairs $x < y$ for which $\sigma(x) > \sigma(y)$.

We break the proof into three claims:

**Claim 1** Permutations with odd cycles cancel out in $det(T)$.

Let $P$ be the set of permutations in $S_n$ that contain at least 1 odd cycle. For each $\sigma \in P$, let $C_\sigma$ be the odd cycle in $\sigma$ with minimum element, and let $\sigma'$ be $\sigma$ with $C_\sigma$ reversed.

For example, if $\sigma = (1,5)(2,3,4)(6,7,8)$, then $\sigma' = (1,5)(4,3,2)(6,7,8)$.

$sign(\sigma) = sign(\sigma')$, so it follows that

$$\prod_{i=1}^{n} T(i,\sigma(i)) = -\prod_{i=1}^{n} T(i,\sigma'(i))$$

because the odd cycle $C_\sigma = (z_1,\ldots,z_r)$ leads to entries $T(z_1,z_2)\cdot\dots\cdot T(z_r,z_1)$ in the first term and entries $T(z_2,z_1)\cdot\dots\cdot T(z_1,z_r) = (-1)^r \cdot T(z_1,z_2)\cdot\dots\cdot T(z_r,z_1)$.

**Claim 2** Any $\sigma$ with only even cycles corresponds to a perfect matching.

For any even cycle $C = (z_1,\ldots,z_{2r})$ in $\sigma$, we can select edges $(z_1,z_2),(z_3,z_4),\ldots,(z_{2r-1},z_{2r})$ to be in the matching. These edges are node-disjoint and cover all the vertices.

Note that Claim 1 in conjunction with Claim 2 demonstrates that whenever $det(T) \neq 0$, there is a $\sigma$ with only even cycles so a perfect matching exists.

**Claim 3** If $G$ has a perfect matching, then $det(T) \neq 0$.

Say $(a_1,b_1),\ldots,(a_{n/2},b_{n/2})$ is a perfect matching. Consider the permutation $\sigma = (a_1,b_1)\ldots(a_{n/2},b_{n/2})$. Then

$$\prod_{i=1}^{n} T(i,\sigma(i)) = \prod_{i=1}^{n/2} T(a_i,b_i)T(b_i,a_i) = \prod_{i=1}^{n/2} -(x_{a_i,b_i})^2$$

No other permutation has the same variables, so this term can't cancel out. It follows that $det(T) \neq 0$.

□

The determinant $\det(T)$ is an $n^2$-variate polynomial of degree $n$ and therefore can be expensive to compute.

**Theorem 1.2** (Lovasz). *If we pick values $v_{ij}$ for each $x_{ij}$ uniformly at random from $\{1,\ldots,n^2\}$ and let $T(\{v_{ij}\})$ be the matrix obtained from $T$ by these substitutions, then $\det(T(\{v_{ij}\})) \neq 0$ iff $\det(T) \neq 0$, with high probability.*

This gives a polynomial time algorithm for Perfect Matching that works with high probability. To prove this theorem we use:

**Lemma 1.1** (Schwartz-Zippel). *Let $P$ be a non-zero polynomial over $\{x_1,\ldots,x_N\}$ of degree $d$ over a field $\mathbb{F}$. If we pick values $v_1,\ldots,v_N$ randomly from a finite set $S \subseteq \mathbb{F}$ and let $P(\{v_i\})$ be the value obtained by setting $x_1 = v_1,\ldots,x_N = v_N$ in $P$, then $P(\{v_i\}) \neq 0$ with probability at least $1 - \frac{d}{|S|}$.*

For $\det(T)$ we have $\deg(\det(T)) = n$ and therefore it is enough to pick $|S| = n^2$. However, if we work over $\mathbb{Z}$ the entries of this determinant could be very large and we only get a running time of $O(n^{\omega+1})$. Instead, pick a prime $p \geq n^3$ and work over $\mathbb{Z}_p$. If $G$ has a perfect matching $M$ then the polynomial $\det(T) \mod p$ contains the non-zero term $f_{\sigma_M}$ and is therefore a non-zero polynomial and we can apply the Schwartz-Zippel lemma to check whether the determinant is zero in $O(n^\omega)$ time.

# 2 Finding the matching

The above algorithm tells us in $O(n^\omega)$ time whether the graph contains a perfect matching. In the rest of this lecture (and the next one) we will discuss algorithms that can find the perfect matching for us.

There is a simple $O(n^{\omega+2})$ solution: for every edge $e \in E$, remove it from the graph and check if there is still a perfect matching in $O(n^\omega)$ time. If the graph does not contain a perfect matching any more, put the edge back and move on to the next edge, otherwise leave the edge out of the graph. What we get in the end is a graph with $n/2$ edges that contains a perfect matching and we're done.

Today we will see an $O(n^{\omega+1})$ algorithm and next week we'll see an $O(n^\omega)$ one.

## 2.1 The Rabin-Vazirani Algorithm

We will prove this thorem.

**Theorem 2.1** (Rabin-Vazirani). *A perfect matching can be found in $O(n^{\omega+1})$ time.*

Consider Algorithm 1.

---

**Algorithm 1:** RV($G$)

---

$T \leftarrow T(\{v_{ij}\})$: a random substitution of the Tutte matrix modulo a large enough prime;
**if** $\det(T) = 0$ **then**
    return no perfect matching;
Set $M = \emptyset$;
**while** $|M| < n/2$ **do**
    Compute $N = T^{-1}$ in $O(n^\omega)$ time;
    Find $j$ such that $N[1, j] \neq 0$ and $(1, j) \in E$;
    $M \leftarrow M \cup \{(1, j)\}$;
    $T \leftarrow T_{\{1,j\},\{1,j\}}$ i.e. remove rows 1 and $j$ and columns 1 and $j$ from $T$;

---

We use the notation $T_{X,Y}$ for subsets $X, Y \subseteq [n]$ to denote the matrix obtained from $T$ by removing the rows indexed by $X$ and columns indexed by $Y$.

Clearly, the algorithm performs $O(n)$ computations that take $O(n^\omega)$ time and therefore runs in $O(n^{\omega+1})$ time. In fact, the algorithm is choosing some $e = (1, j)$ in some perfect matching and recursing on $G \setminus \{1, j\}$. We show the correctness below.

Recall the adjoin formula:

$$T^{-1}[i, j] = (-1)^{i+j} \cdot \frac{\det(T_{\{i\},\{j\}})}{\det(T)},$$

and therefore in the algorithm we have that $N[1, j] \neq 0$ iff $\det(T_{\{i\},\{j\}}) \neq 0$.

By the definition of the determinant:

$$\det(T) = \sum_{j=1}^{n} (-1)^{1+j} \cdot T[1, j] \cdot \det(T_{\{1\}\{j\}}),$$

and therefore if $\det(T) \neq 0$ then there exists $j \in [n]$ such that $T[1, j] \cdot \det(T_{\{1\}\{j\}}) \neq 0$ and therefore $(1, j) \in E$ and $\det(T_{\{1\}\{j\}}) \neq 0$. Therefore, to show the correctness of the algorithm, it is enough to show that the latter also implies that $\det(T_{\{1,j\}\{1,j\}}) \neq 0$ (i.e. when removing $\{1, j\}, \{1, j\}$ instead of just $\{1\}\{j\}$.)

To prove this, we need to use properties of the Tutte matrix. Note that $T$ is a *skew symmetric* matrix: $T = -T^t$.

**Proposition 1.** *Let $A$ be an $n \times n$ skew symmetric matrix, then:*

1. *$A^{-1}$ is skew symmetric.*

2. *If $n$ is odd, then* $\det(A) = 0$.

3. *(Frobenius) Let $Y \subseteq [n]$ s.t. $|Y| = rank(A)$ and the column rank of $A[[n], Y]$ is $rank(A)$, then* $\det(A[Y, Y]) \neq 0$.

Proof of 2: $\det(A) = \det(-A^t) = (-1)^n \det(A)$. We will use 3 without proof.

**Lemma 2.1.** *If* $\det(T_{\{1\},\{j\}}) \neq 0$ *then* $\det(T_{\{1,j\},\{1,j\}}) \neq 0$.

*Proof.* Assume without loss of generality that $j = 2$. By property 2 we know that $A = T_{\{1\},\{1\}} = 0$, so its rank is at most $n - 2$. By our assumption, $\det(T_{\{1\},\{2\}}) \neq 0$ so $\det(T_{\{1\},\{2\}})$ has rank $n - 1$. Therefore the column rank of $T_{\{1\},\{1,2\}}$ is $n - 2$ and the rank of $A$ is $n - 2$. $A$ is skew-symmetric, so it follows from the Frobenius property that for $Y = \{3, \ldots, n\}$, $det(A[Y, Y] = det(T_{\{1\},\{2\}} \neq 0$. ☐